MST121 Chapter B3

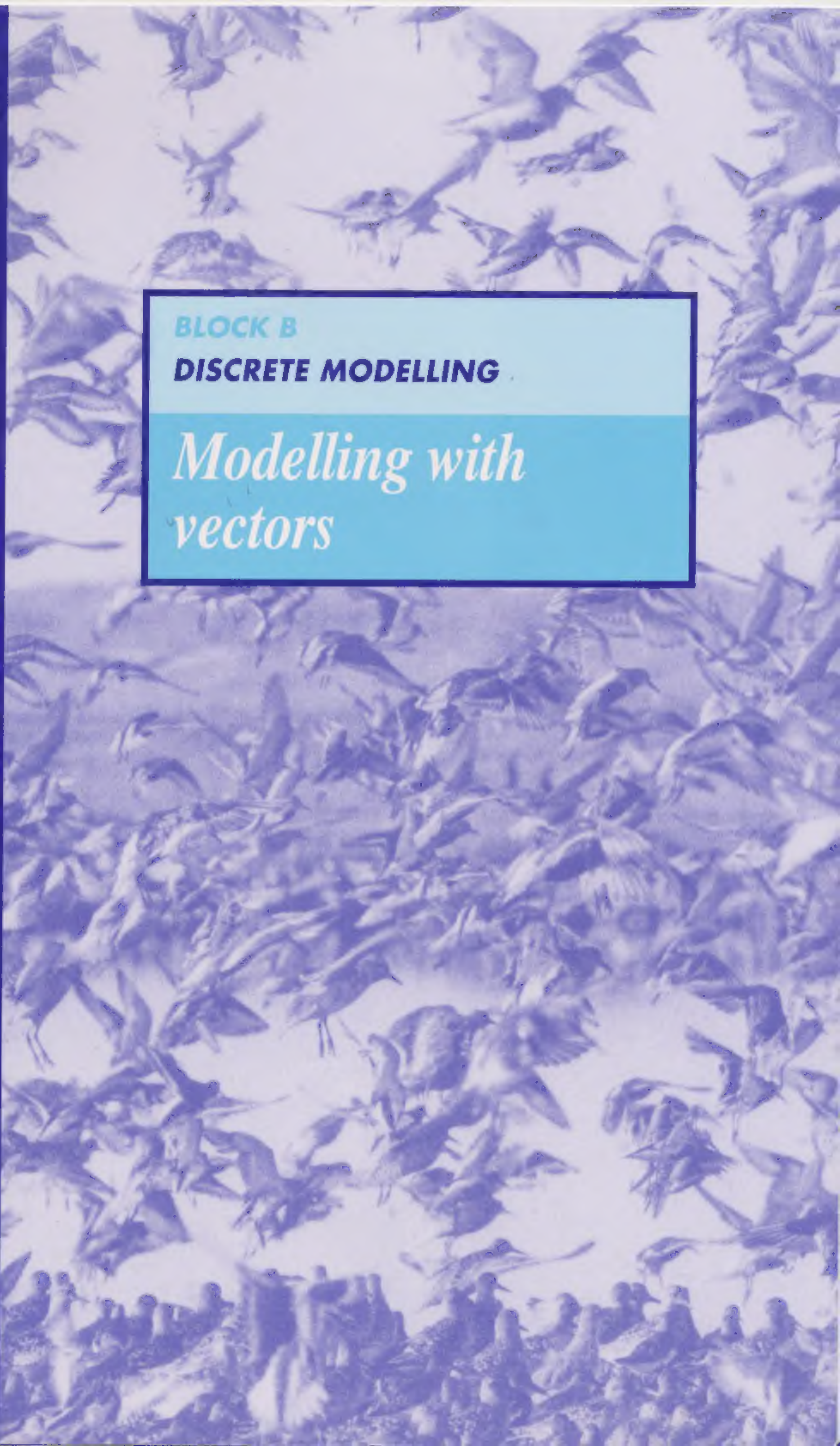
The Open University

A first level interdisciplinary course

Using Mathematics

CHAPTER

B3



BLOCK B

DISCRETE MODELLING

Modelling with vectors

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Modelling with vectors

Prepared by the course team

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Contents

Study guide

There are five sections in this chapter. They are intended to be studied consecutively in five study sessions. Each session requires two to three hours.

The pattern of study for each session might be as follows.

Study session 1: Section 1.
Study session 2: Section 2.
Study session 3: Section 3.
Study session 4: Section 4.
Study session 5: Section 5.

Subsection 4.1 contains no activities.
Subsection 5.1 requires the use of an audio cassette player. Subsection 5.2 will not be assessed. Note that no use of the computer is required in this chapter, although Computer Book B suggests some uses to which the computer might be put in the context of the material in this text.

You will need to make much use of your calculator in this chapter to evaluate expressions involving degrees, so remember to set your calculator to degree mode. Also remember to use unrounded calculated values in the later stages of a calculation.

The optional Video Band B(iv) *Algebra workout – working with vectors* could be viewed at any stage during your study of this chapter.

Introduction

In modelling the real world, variables are chosen to represent measurable quantities. In many cases, the variable concerned takes values which are real numbers. For example, length, area, time and temperature can all be represented in this way. Variables of this type are called *scalars*, and you should be familiar with how to manipulate them from previous work.

By contrast, the values of some measurable quantities cannot be represented adequately by just real numbers. For example, the specification of a wind velocity requires knowledge of both how fast the wind blows *and* the direction in which it blows. A quantity like this, which involves both a magnitude and a direction, is called a *vector*. Other examples of vector quantities are *displacement* (the position of one point relative to another point), *acceleration* and *force*.

In Chapter B2, you saw the word 'vector' used to describe a column of numbers. On the face of it, this is not the same as a quantity with both magnitude and direction. However, it turns out that the 'column of numbers' and the 'quantity with magnitude and direction' are two equivalent representations of the same underlying vector idea.

Section 1 reviews what you saw of vectors as columns of numbers in Chapter B2. It then gives an alternative representation for these columns as arrows, which provide a *geometric form* of vectors. Each arrow is characterised by a magnitude (its length) and the direction in which it points. The operations of vector addition and scalar multiplication of vectors, which were introduced in Chapter B2 for vectors in column form, can be described in a consistent way for vectors in geometric form. This form for vectors is useful for visualisation, but may be less so for precise calculation. Therefore a third representation for vectors is introduced, the *component form*. This features the same numbers (components) that appear in the column form of the vector, but makes explicit which direction in space is associated with each of these components, by use of the *Cartesian unit vectors*.

Section 2 explains how to convert vectors in column or component form to the corresponding (geometric) form in terms of magnitude and direction, and vice versa. It then looks at applications of the vector ideas introduced so far to situations that involve *displacement* and *velocity*.

Section 3 develops some further trigonometry, by establishing the *Sine Rule* and *Cosine Rule* for solving triangles. These are applied to problems that concern displacement and velocity, where the two rules provide an alternative to the approach of Section 2.

As you may recall from Chapter A2, Subsection 3.2, *solving a triangle* means finding all angles and side lengths of the triangle.

The rest of the text introduces the subject of *statics*, by showing how vectors can be used to model the *forces* (pushes or pulls) acting on an object that does not move. Section 4 describes what a force is, and discusses the examples of *weight*, *tension* and *normal reaction*. It then gives a mathematical statement of what a balance of forces entails.

Section 5 illustrates alternative strategies (based on use of the *triangle of forces* or components) for solving problems that involve three forces. Where there are more than three forces, the component method is often the only practical option. The chapter concludes with a (non-assessed) look at the force of *friction*.

1 Vectors and geometry

In this section the material in Chapter B2 about vectors as columns of numbers is reviewed, and then the alternative geometric and component forms of vectors are introduced. In each case, the operations of vector addition and scalar multiplication of vectors are described.

1.1 Vectors in column form

See Chapter B2, Subsection 1.2. The components of vectors like these were used to represent the sizes of human subpopulations (in millions), or the amounts of water (in litres) being input to or output from the nodes in a network of pipes.

In Chapter B2, you saw the word *vector* used to describe a column of numbers, such as

$$\begin{pmatrix} 10.92 \\ 46.49 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0.3 \\ 0.1 \\ 0.6 \end{pmatrix}.$$

Following the introduction of matrices, it was pointed out that 'vector' is simply a shorter name for 'matrix with one column'. The numbers appearing in the column are called the *components* of the vector, so the two vectors above have respectively two and three components. In this text, we shall consider only two-component vectors, but very similar results apply for three-component vectors.

The use of bold type also applies to vectors in the other forms to be considered later. However, in Sections 4 and 5, upper-case bold letters, such as **F**, are used to describe vectors which represent forces.

In print, vectors are represented by bold lower-case letters, such as **a**, but when writing by hand, the convention for denoting vectors is to underline the symbol with either a straight line, $\underline{a}$, or with a wavy line, $\underset{\sim}{a}$.

If the components of the two-component vector **a** are not given numerically, then they may be denoted by a_1 and a_2, as follows:

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

In each case, these are special cases of the corresponding operations for matrices, as defined in Chapter B2, Subsection 2.2.

Operations of *addition* and *scalar multiplication* may be defined for vectors in column form.

Vectors are added according to the following rule.

Addition of vectors (in column form)

The sum of two vectors with the same number of components is formed by adding the corresponding components. For two-component vectors, this means that

$$\text{if } \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \text{ then } \mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}.$$

For example, we have

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2+3 \\ 3+(-1) \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

Since this operation involves the addition of corresponding components, there is no need to restrict it to just two vectors. We can add together any number of vectors, using the normal rules of algebra to find the sums of corresponding components.

Example 1.1 Adding vectors (in column form)

Let $\mathbf{a} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$. Find the vector $\mathbf{a}+\mathbf{b}+\mathbf{c}$.

Solution

The sum of the three given vectors is

$$\mathbf{a}+\mathbf{b}+\mathbf{c} = \begin{pmatrix} 4+(-3)+2 \\ 1+2+(-2) \end{pmatrix} = \begin{pmatrix} 4-3+2 \\ 1+2-2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Activity 1.1 Adding vectors (in column form)

Find each of the following vector sums.

(a) $\begin{pmatrix} 4 \\ -2 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} 5 \\ 3 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \end{pmatrix}$ (c) $\begin{pmatrix} -7 \\ -4 \end{pmatrix} + \begin{pmatrix} 2 \\ 7 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

Solutions are given on page 58.

The phrase *scalar multiplication* refers to the multiplication of a vector by a real number. The real number is called a *scalar*, in order to emphasise that it is different in kind from the vector which it multiplies.

The terms *scalar* and *vector*, in their modern senses, both appeared first in an 1844 paper by the Irish mathematician Sir William Rowan Hamilton (1805–1865).

Scalar multiplication of a vector (in column form)

The scalar multiple of a vector $\mathbf{a}$ by a real number (scalar) k, denoted by $k\mathbf{a}$, is formed by multiplying each component of the vector $\mathbf{a}$ by k. For a two-component vector, this means that

$$\text{if } \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \text{ then } k\mathbf{a} = \begin{pmatrix} ka_1 \\ ka_2 \end{pmatrix}.$$

For example, we have

$$3\begin{pmatrix} 4 \\ -5 \end{pmatrix} = \begin{pmatrix} 12 \\ -15 \end{pmatrix}.$$

The scalar k which multiplies the vector $\mathbf{a}$ may be positive, negative or zero. In particular, if $k=-1$, then we have the scalar multiple $(-1)\mathbf{a}$, which is usually abbreviated to $-\mathbf{a}$. Similarly, $(-2)\mathbf{a}$ is written as $-2\mathbf{a}$, and so on.

Activity 1.2 Scalar multiplication of a vector (in column form)

Let $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Find each of the following scalar multiples of $\mathbf{a}$.

(a) $4\mathbf{a}$ (b) $-\mathbf{a}$ (c) $\frac{1}{2}\mathbf{a}$

Solutions are given on page 58.

This is a special case of the rule for matrix subtraction, as defined in Chapter B2, Subsection 2.2.

The material in the rest of this subsection goes beyond what was covered in Chapter B2.

The *subtraction* of vectors is defined in terms of vector addition and scalar multiplication, according to the rule

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-1)\mathbf{b}.$$

If a vector $\mathbf{a}$ is subtracted from itself, then we obtain

$$\mathbf{a} - \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + (-1)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 - a_1 \\ a_2 - a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The vector on the right-hand side, with both components zero, is called the **zero vector** and is denoted by $\mathbf{0}$ (a zero in bold type). It arises also if any vector $\mathbf{a}$ is multiplied by the scalar zero, since

$$0\mathbf{a} = 0\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \times a_1 \\ 0 \times a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}.$$

For any vector $\mathbf{a}$, we have $\mathbf{a} + \mathbf{0} = \mathbf{a}$. The importance of the zero vector in one modelling context will become apparent towards the end of the chapter.

The vector operations of addition, subtraction and scalar multiplication can be applied in combination. For example, we have

$$\begin{pmatrix} -7 \\ 4 \end{pmatrix} - 3\begin{pmatrix} 2 \\ -1 \end{pmatrix} + 8\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -7 - 3 \times 2 + 8 \times 1 \\ 4 - 3 \times (-1) + 8 \times 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}.$$

Activity 1.3 Combining vector operations (in column form)

Write each of the following as a single vector.

(a) $2\begin{pmatrix} 6 \\ -3 \end{pmatrix} - 7\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 5\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

(b) $a_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, where a_1, a_2 are any scalars.

Solutions are given on page 58.

The result of Activity 1.3(b) is worth emphasising. It shows that any vector $\mathbf{a}$ can be represented as a sum of scalar multiples of the two vectors

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where the two scalar multipliers are just the respective components of $\mathbf{a}$. In other words,

$$\text{if } \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \text{ then } \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}.$$

This result will be referred to later in the section.

1.2 Vectors in geometric form

So far you have seen vectors (columns of numbers) used to model amounts of water entering or leaving the nodes of a pipe network, or the numbers of individuals in human subpopulations. Another type of situation in which vectors can be used as a model is now introduced.

Suppose that we wish to describe the position of Leeds relative to that of Bristol. One way of doing so is to note that Leeds is approximately 77 kilometres to the East of Bristol and 286 kilometres to the North of Bristol, and to store this information in column form as

$$\begin{pmatrix} 77 \\ 286 \end{pmatrix},$$

where the first component of the vector denotes 'distance to the East' and the second denotes 'distance to the North', both in kilometres. This column, together with its interpretation, provides complete information about where Leeds is located relative to Bristol, and permits us to place them relative to each other on a 'map', as illustrated in Figure 1.1(a). However, it is not the only form in which the data could be presented, and for many purposes it would not be the most convenient. If you wanted to fly an aeroplane between these two cities, then you would need to know:

As a simplifying assumption, the departures from flatness that occur on the surface of the Earth are ignored throughout this chapter.

- how far (in a straight line) Leeds is from Bristol;
- in what direction Leeds lies from Bristol.

It is possible to obtain these items of information from the column form given above. In fact, Leeds is about 296 kilometres from Bristol, in a straight line, and Leeds would be reached from Bristol by travelling in the direction which is 15° to the East of due North (which we abbreviate to N 15°E). This description is illustrated in Figure 1.1(b).

You will see later how these items are calculated. Rough values for them can be obtained by drawing a scale diagram such as Figure 1.1(a) and then taking appropriate measurements.

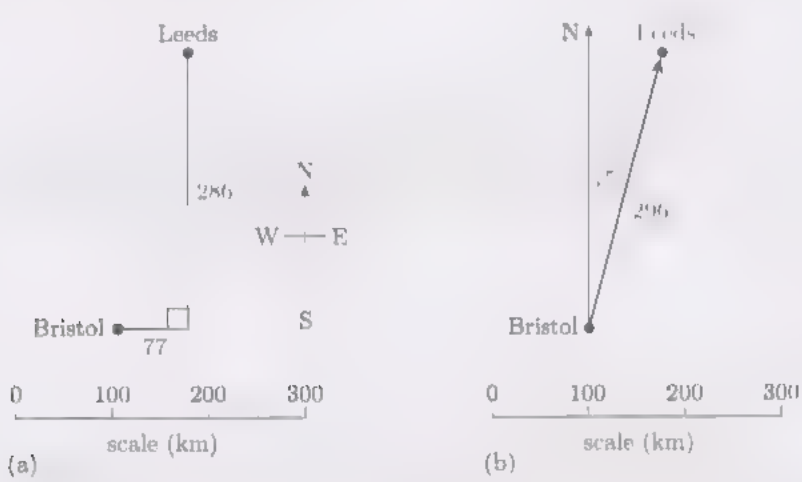


Figure 1.1 Location of Leeds relative to Bristol: (a) in terms of distances East and North; (b) in terms of straight-line distance and direction

The arrow in Figure 1.1(b) represents the vector whose column form was used to draw Figure 1.1(a). The vector (arrow) has *magnitude* 296 km and *direction* N 15°E. These quantities specify the *geometric form* of the vector.

The geometric form of vector is often introduced first in teaching texts, with the informal definition being that a vector is a 'quantity that has both magnitude and direction'.

In describing the vector represented in Figure 1.1(b), we used a distance scale based upon kilometres and took East and North as reference directions. More generally, we can choose x- and y-axes to provide reference scales and directions. Then any vector **a** with two components can be represented by an arrow in the (x, y)-plane. For example, Figure 1.2 (overleaf) shows arrows drawn to represent the four vectors

A vector with three components can similarly be represented by an arrow in space.

$$\mathbf{p} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}$$

Note that each of these arrows can be placed anywhere in the plane, since a vector has magnitude and direction but not location

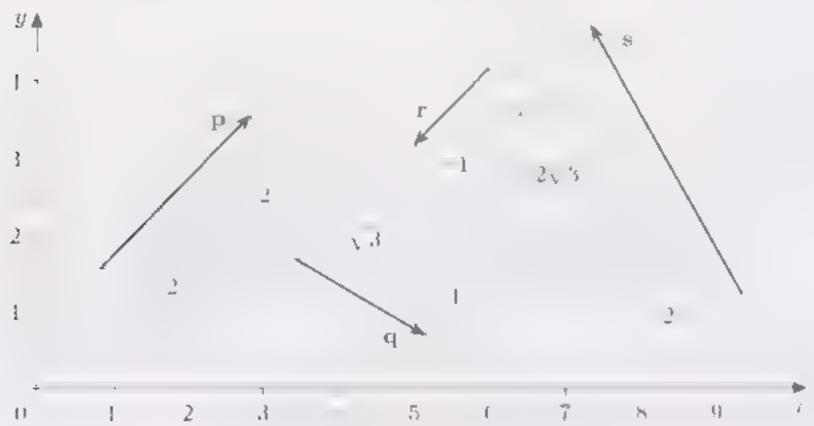


Figure 1.2 Arrows representing four vectors

This figure illustrates that, for any vector $\mathbf{a}$, the signs of the components a_1 and a_2, as well as their sizes, determine the direction of the arrow. Thus the arrow points:

- in a direction for which x increases if $a_1 > 0$, and in a direction for which x decreases if $a_1 < 0$;
- in a direction for which y increases if $a_2 > 0$, and in a direction for which y decreases if $a_2 < 0$.

Activity 1.4 Sketching arrows to represent vectors

Sketch arrows in the (x, y)-plane to represent the vectors

$$\mathbf{t} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Solutions are given on page 58.

The **magnitude** of a vector is the length of a corresponding arrow, and the notation $|\mathbf{a}|$ is used for the magnitude of the vector $\mathbf{a}$. As Figure 1.2 suggests, the arrow length can be found from the components by using Pythagoras' Theorem, which gives the formula

This is sometimes called the *modulus* of $\mathbf{a}$. You saw the modulus notation $|x|$ for a real number x in Chapter A3. The extension of the notation to vectors retains the property that the value of the modulus is always non-negative.

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}.$$

For example, the magnitude of the vector $\mathbf{q}$ in Figure 1.2 is

$$|\mathbf{q}| = \sqrt{q_1^2 + q_2^2} = \sqrt{(\sqrt{3}\,)^2 + (-1)^2} = \sqrt{4} = 2.$$

Activity 1.5 Finding vector magnitudes

Find the magnitude of each of the vectors $\mathbf{p}$, $\mathbf{r}$ and $\mathbf{s}$ shown in Figure 1.2.

Solutions are given on page 58.

The **direction** of a vector may be described by the angle θ that the corresponding arrow makes with the positive x-direction. This angle is usually chosen to lie within the range $-180° < \theta \leq 180°$, where a positive value of θ corresponds to an anticlockwise rotation from the positive x-direction, and a negative value to a clockwise rotation. This is illustrated for some values of θ in Figure 1.3.

In radians, this range is $-\pi < \theta \leq \pi$.

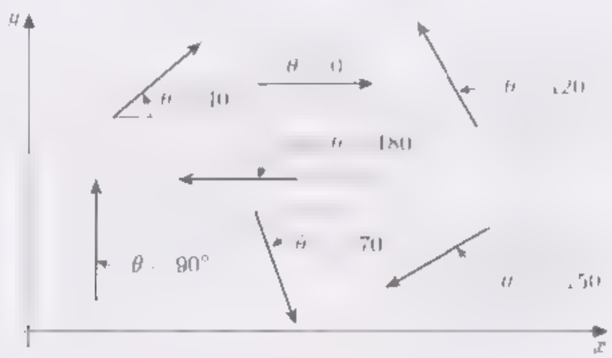


Figure 1.3 How the value of θ describes direction

Using this convention, we can find a unique value of θ for each arrow drawn in the plane. For example, consider the vector $\mathbf{q}$ for which an arrow is drawn in Figure 1.2. The arrow lies along the hypotenuse of a right-angled triangle for which the other two sides have lengths $\sqrt{3}$ and 1. The angle that the arrow makes with the positive x-direction can be seen to have size 30°, since $\tan(30°) = 1/\sqrt{3}$. However, the arrow points 'to the right and down', so the direction of $\mathbf{q}$ is given by $\theta = -30°$.

Activity 1.6 Finding vector directions

Using the facts that $\tan(45°) = 1$ and $\tan(60°) = \sqrt{3}$, find the direction (value of θ) for each of the vectors $\mathbf{p}$, $\mathbf{r}$ and $\mathbf{s}$ shown in Figure 1.2.

Solutions are given on page 58.

We shall look further at how to find the directions of vectors from their components in Section 2. For the moment, note that any non-zero vector $\mathbf{a}$ may be described completely *either* in terms of its components (column form) *or* in terms of its magnitude $|\mathbf{a}|$ and direction θ (**geometric form**), as illustrated in Figure 1.4. The zero vector, $\mathbf{0}$, has magnitude 0 and no associated direction.

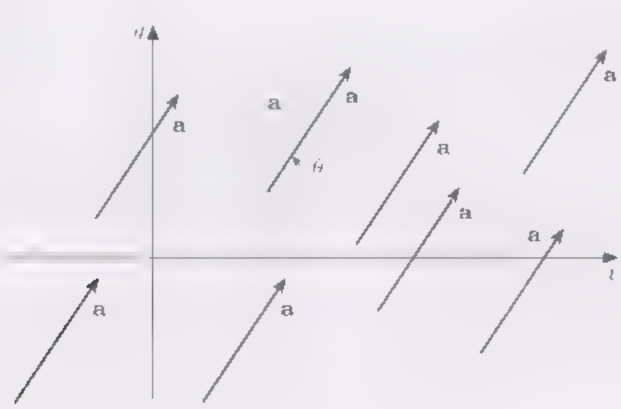


Figure 1.4 Arrows representing a vector in geometric form

Figure 1.4 shows several copies of the arrow which represents a vector **a**, at different positions in the plane. This figure emphasises that magnitude and direction are both determining features of a vector, but position within the plane is not. In other words, any two arrows which have both the same length and the same direction represent the same vector.

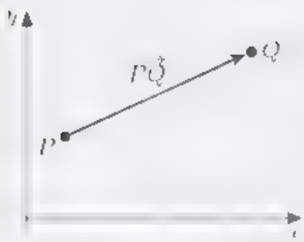


Figure 1.5 Arrow for the displacement vector $\overrightarrow{PQ}$

The arrow in Figure 1.1(b) represents the displacement from Bristol to Leeds. The same displacement prescription (move 296 km in a direction N 15°E) could also be applied to any starting point other than Bristol.

It is sometimes convenient to specify a vector with reference to particular points of the (x, y)-plane. If P and Q are any two points in the (x, y)-plane, then an arrow can be drawn with its tail at P and its tip at Q. This arrow represents the **displacement** (**vector**) from P to Q, for which the notation $\overrightarrow{PQ}$ is often used (see Figure 1.5). The vector **a** that is defined by the magnitude and direction of the displacement vector $\overrightarrow{PQ}$ may be represented by an arrow of the same magnitude and direction placed anywhere in the plane. Thus a displacement vector may be regarded as specifying a displacement (or translation) for any point in the plane, while the notation $\overrightarrow{PQ}$ emphasises its effect on the particular point P, which is displaced to Q.

The column form of the displacement vector $\overrightarrow{PQ}$ may be written down in terms of the coordinates (p_1, p_2) of the point P and (q_1, q_2) of the point Q. In fact, as Figure 1.6 illustrates, we have

$$\overrightarrow{PQ} = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \end{pmatrix}.$$

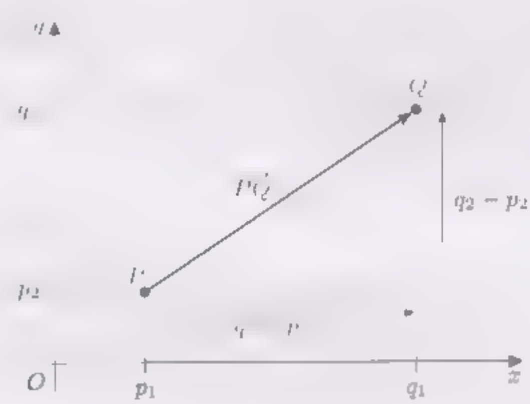


Figure 1.6 Displacement vector $\overrightarrow{PQ}$ in terms of coordinates of P and Q

Note that in the particular case when the tail P of the arrow is at the origin, $O\,(0, 0)$, we have

$$\overrightarrow{OQ} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

This is known as the **position vector** of the point Q, and its components coincide with the corresponding coordinates of Q.

Vector addition and scalar multiplication (in geometric form)

To complete an introduction to the geometric form of vectors, it remains to explain the geometrical meaning of vector *addition* and *scalar multiplication*. The first of these can be approached with reference to the displacement vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ shown in Figure 1.7(a).

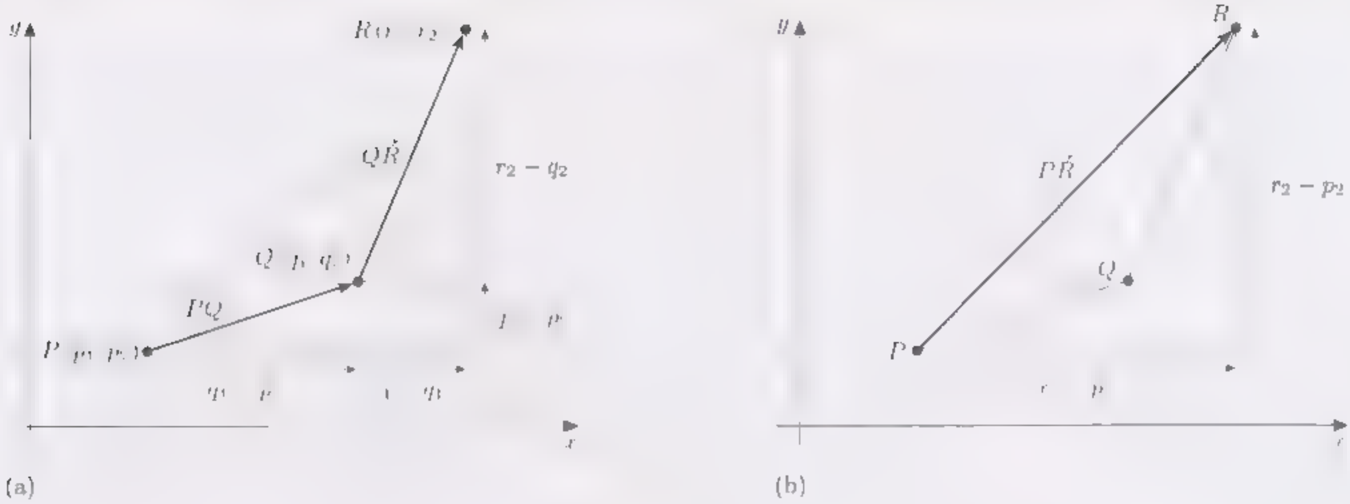


Figure 1.7 Adding the displacement vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$

As explained above, these vectors can be expressed in terms of the coordinates of the points P, Q and R. Using the rule for the addition of vectors in column form from Subsection 1.1, we have

$$\overrightarrow{PQ} + \overrightarrow{QR} = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \end{pmatrix} + \begin{pmatrix} r_1 - q_1 \\ r_2 - q_2 \end{pmatrix} = \begin{pmatrix} r_1 - p_1 \\ r_2 - p_2 \end{pmatrix} = \overrightarrow{PR}.$$

This demonstrates that the sum of the displacement vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ is the displacement vector $\overrightarrow{PR}$, whose arrow lies along the third side of the triangle PQR (see Figure 1.7(b)). The same *Triangle Rule* holds for any pair of vectors in geometric form.

Triangle Rule for addition of vectors

To find in geometric form the sum $\mathbf{a} + \mathbf{b}$ of any two vectors $\mathbf{a}$ and $\mathbf{b}$.

1. Choose any point P in the plane.
2. Draw an arrow to represent $\mathbf{a}$, with tail at P and tip at Q, say.
3. Draw an arrow to represent $\mathbf{b}$, with tail at Q and tip at R, say.
4. Draw the arrow with tail at P and tip at R, to complete the triangle PQR.

Then this last arrow represents the vector $\mathbf{a} + \mathbf{b}$. The construction is illustrated in Figure 1.8(a).

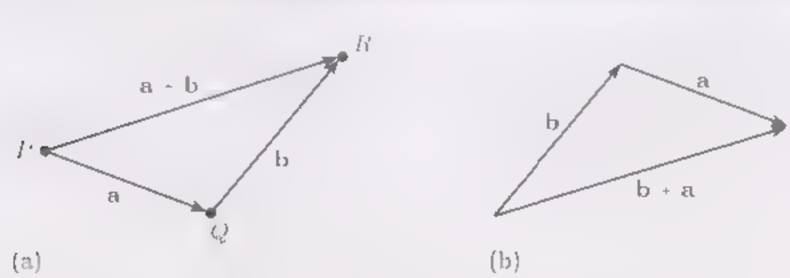


Figure 1.8 Adding the vectors $\mathbf{a}$ and $\mathbf{b}$ by the Triangle Rule

The sum vector $\mathbf{a} + \mathbf{b}$ is called the **resultant** of $\mathbf{a}$ and $\mathbf{b}$. Note that the Triangle Rule produces the same resultant if the two vectors are added in the opposite order, as is to be expected from the corresponding addition rule in column form. Thus the triangle in Figure 1.8(a) could be replaced by that in Figure 1.8(b), where the vectors $\mathbf{a}$ and $\mathbf{b}$ are the same in each case.

This suggests an alternative (but equivalent) construction for the geometric form of a vector sum, which is known as the *Parallelogram Rule*.

Parallelogram Rule for addition of vectors

To find in geometric form the sum $\mathbf{a} + \mathbf{b}$ of any two vectors $\mathbf{a}$ and $\mathbf{b}$.

1. Choose any point P in the plane.
2. Draw an arrow to represent $\mathbf{a}$, with tail at P and tip at Q, say.
3. Draw an arrow to represent $\mathbf{b}$, with tail at P and tip at S, say.
4. Complete the parallelogram $PQRS$, and draw the arrow with tail at P and tip at R.

Then this last arrow represents the vector $\mathbf{a} + \mathbf{b}$. The construction is illustrated in Figure 1.9.

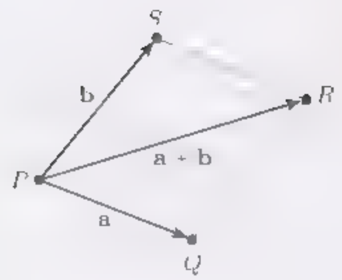


Figure 1.9 Adding the vectors $\mathbf{a}$ and $\mathbf{b}$ by the Parallelogram Rule

Activity 1.7 Finding vector sums geometrically

Throughout this chapter, the Triangle Rule is used in preference to the Parallelogram Rule.

For the vectors represented in Figure 1.2, use the Triangle Rule to sketch arrows to represent the following sums.

(a) $\mathbf{p} + \mathbf{q}$ (b) $\mathbf{r} + \mathbf{s}$

Solutions are given on page 59.

Comment

Sketching an arrow to represent a vector sum like this will provide only rough estimates for the magnitude and direction of the resultant. You will see how to obtain accurate values for these in Sections 2 and 3.

The effect of scalar multiplication on the geometric form of a vector can also be deduced from the corresponding rule for vectors in column form. As recalled in Subsection 1.1, the effect of multiplication by the scalar k is given by:

$$\text{if } \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \text{ then } k\mathbf{a} = \begin{pmatrix} ka_1 \\ ka_2 \end{pmatrix}.$$

If k is *positive*, then each component a_1, a_2 is multiplied by this positive factor. As a result, the direction of $k\mathbf{a}$ is the same as that of $\mathbf{a}$, since the two right-angled triangles shown in Figure 1.10(a) are similar. For the same reason, the magnitude of $k\mathbf{a}$ is k times that of $\mathbf{a}$. On the other hand, if k is *negative*, then we again have similar right-angled triangles (see Figure 1.10(b)), but now each component of $k\mathbf{a}$ is reversed in sign. This has the effect of making the arrow for $k\mathbf{a}$ point in the opposite direction to that for $\mathbf{a}$, while the magnitude of $k\mathbf{a}$ (which must be non-negative) is $|k|$ times that of $\mathbf{a}$.

Recall from Chapter A3 that

$$|k| = \begin{cases} k, & \text{if } k \geq 0, \\ -k, & \text{if } k < 0. \end{cases}$$

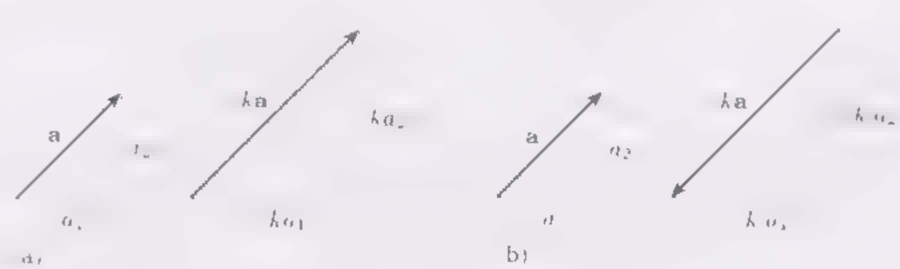


Figure 1.10 The vectors $\mathbf{a}$ and $k\mathbf{a}$: (a) $k > 0$; (b) $k < 0$

Scalar multiplication of a vector (in geometric form)

If $\mathbf{a}$ is a vector and k is a real number, then the scalar multiple $k\mathbf{a}$ has:

- magnitude $|k\mathbf{a}| = |k||\mathbf{a}|$;
- direction which is the same as that of $\mathbf{a}$ if $k > 0$, or opposite to that of $\mathbf{a}$ if $k < 0$.

Note the different uses of the modulus notation here: $|k|$ is the modulus of a real number, while $|\mathbf{a}|$ and $|k\mathbf{a}|$ are the magnitudes of vectors.

Recall that if $k = 0$, then $k\mathbf{a} = \mathbf{0}$, which has magnitude 0 and no direction.

This rule for scalar multiplication is illustrated in Figure 1.11.

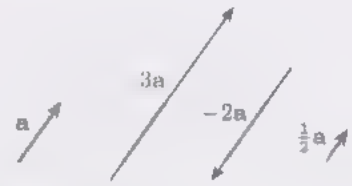


Figure 1.11 Scalar multiples of a vector $\mathbf{a}$

Activity 1.8 Finding scalar multiples geometrically

For the vectors represented in Figure 1.2, sketch arrows to represent the following scalar multiples.

(a) $3\mathbf{r}$ (b) $-\mathbf{s}$ (c) $-2\mathbf{q}$

Solutions are given on page 59.

The rule for vector *subtraction* in geometric form follows from those for addition and scalar multiplication, using the same formula as applied for column form; that is,

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-1)\mathbf{b}.$$

Hence the vector subtraction $\mathbf{a} - \mathbf{b}$ is performed geometrically by using either the Triangle or Parallelogram Rule to add $\mathbf{a}$ and $-\mathbf{b}$, where the arrow for $-\mathbf{b}$ has the same size as that for $\mathbf{b}$ but the opposite direction. For example, for the vectors in Figure 1.2, the constructions of arrows to represent $\mathbf{p} - \mathbf{q}$ and $\mathbf{s} - \mathbf{r}$ are shown in Figure 1.12.

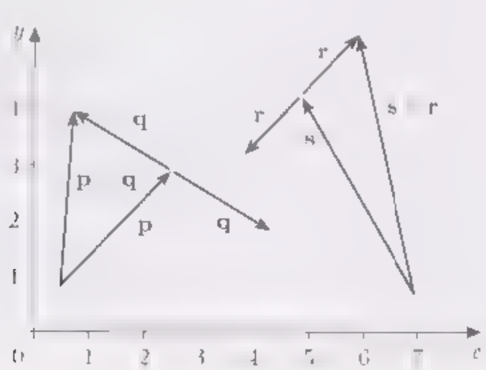


Figure 1.12 Examples of vector subtraction using the Triangle Rule

Activity 1.9 A sum and difference of vectors

The vector $\mathbf{a}$ has magnitude 3 units and its arrow points in the positive x-direction. A vector $\mathbf{b}$ has magnitude 4 units and its arrow points in the positive y-direction. Draw a diagram to show arrows that represent the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$.

Solutions are given on page 59.

Vectors are important because they have wide modelling applicability. It may not have struck you as remarkable that the rule for vector addition provides what seems to be the natural way to combine two displacements, from P to Q and from Q to R, in order to obtain the resultant displacement from P to R, as in Figure 1.7. It is less obvious that the rules for combining vectors should also mirror the way in which other physical quantities behave, but in many important instances they do. In particular, this is the case for *velocity* and for *force*, and the application of the vector model to these quantities occupies a significant part of the rest of the chapter.

1.3 Vectors in component form

A third form of vectors is now introduced. It involves only a slightly different way of looking at column form.

Any vector whose magnitude is 1 is called a *unit vector*. Of particular interest are the **Cartesian unit vectors**,

$$\mathbf{i}=\begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \mathbf{j}=\begin{pmatrix}0\\1\end{pmatrix}.$$

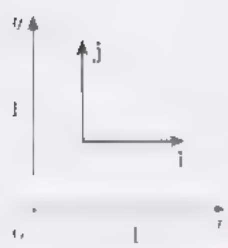


Figure 1.13 The Cartesian unit vectors

Arrows to represent these vectors point respectively in the positive x- and y-directions, as shown in Figure 1.13. As was stated at the end of Subsection 1.1, the vectors $\mathbf{i}$ and $\mathbf{j}$ have the property that

$$\text{if } \mathbf{a}=\begin{pmatrix}a_1\\a_2\end{pmatrix}, \text{ then } \mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}.$$

The expression $a_1\mathbf{i}+a_2\mathbf{j}$ is known as the **component form** of the vector $\mathbf{a}$. The numbers a_1 and a_2 may now be referred to respectively as the **i-component** and **j-component** of $\mathbf{a}$.

a_1 and a_2 are sometimes called the *x-component* and *y-component* of $\mathbf{a}$.

The component form differs only cosmetically from the column form of $\mathbf{a}$, though it tends to be preferred where the Cartesian unit vectors have an interpretation in terms of physical direction as well as measurement. For example, if $\mathbf{i}$ denotes 1 km to the East and $\mathbf{j}$ denotes 1 km to the North, then our earlier column form for the displacement of Leeds from Bristol can be written as $77\mathbf{i}+286\mathbf{j}$.

The close relationship between column and component forms means that we can, without more ado, adapt for component form the rules for vector addition and scalar multiplication that were given in Subsection 1.1.

Addition of vectors (in component form)

The sum of two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ is given by

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j}.$$

Scalar multiplication of a vector (in component form)

The scalar multiple of a vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ by a real number k is given by

$$k\mathbf{a} = ka_1\mathbf{i} + ka_2\mathbf{j}.$$

The rule for vector subtraction follows as before:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-1)\mathbf{b} = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j}.$$

As with column form, these operations for vectors can be applied in combination. For example, for the vectors

$$\mathbf{t} = 5\mathbf{i} + 3\mathbf{j}, \quad \mathbf{u} = -2\mathbf{i} + 7\mathbf{j}, \quad \mathbf{v} = 4\mathbf{i} - 4\mathbf{j},$$

we have

$$\begin{aligned} 3\mathbf{t} - \mathbf{u} - 5\mathbf{v} &= 3(5\mathbf{i} + 3\mathbf{j}) - (-2\mathbf{i} + 7\mathbf{j}) - 5(4\mathbf{i} - 4\mathbf{j}) \\ &= (3 \times 5 - (-2) - 5 \times 4)\mathbf{i} + (3 \times 3 - 7 - 5 \times (-4))\mathbf{j} \\ &= -3\mathbf{i} + 22\mathbf{j}. \end{aligned}$$

Hence the vector $3\mathbf{t} - \mathbf{u} - 5\mathbf{v}$ has $\mathbf{i}$-component -3 and $\mathbf{j}$-component 22.

Activity 1.10 Combining vector operations (in component form)

For the vectors $\mathbf{t}$, $\mathbf{u}$ and $\mathbf{v}$ given above, find the $\mathbf{i}$- and $\mathbf{j}$-components of the vector $-2\mathbf{t} + 3\mathbf{u} + 4\mathbf{v}$.

Solutions are given on page 59.

To conclude the section, note that the expression of a vector in its component form, as $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, is consistent with the Triangle Rule for adding the vectors $a_1\mathbf{i}$ and $a_2\mathbf{j}$; this is shown in Figure 1.14. The arrows for $a_1\mathbf{i}$ and $a_2\mathbf{j}$ are scaled versions of those for the Cartesian unit vectors, and the right-angled triangle that can be formed from them has the arrow for $\mathbf{a}$ along its hypotenuse.

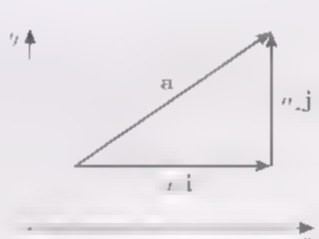


Figure 1.14 Component form is consistent with Triangle Rule for vector addition

Summary of Section 1

This section has reviewed or introduced:

- scalars (another name for real numbers);
- vectors in column form, $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, where a_1, a_2 are the components of $\mathbf{a}$;
- vectors in component form, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, where $\mathbf{i}, \mathbf{j}$ are the Cartesian unit vectors, and a_1, a_2 are respectively the **i**-component and **j**-component of $\mathbf{a}$;
- vectors $\mathbf{a}$ in geometric form, represented by arrows, with magnitude $|\mathbf{a}$ and direction θ;
- the algebraic rules for addition, scalar multiplication and subtraction of vectors in column or component form;
- the corresponding rules for vectors in geometric form, including the Triangle Rule (or, equivalently, the Parallelogram Rule) to obtain the resultant (sum) of two vectors;
- the zero vector, $\mathbf{0}$;
- displacement vectors, denoted by $\overrightarrow{PQ}$, and the expression of their components in terms of the coordinates of the points P and Q.

Exercises for Section 1

Exercise 1.1

Let $\mathbf{a} = -7\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = 5\mathbf{i} - 2\mathbf{j}$ and $\mathbf{c} = 3\mathbf{j}$.

(a) Write down the column form for each of these vectors.

(b) Find the component form of $\mathbf{a} + 3\mathbf{b} - \mathbf{c}$, and specify the **i**- and **j**-components of this vector.

Exercise 1.2

Let $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j}$. Sketch arrows to represent each of the vectors

$$\mathbf{a}, \quad \mathbf{b}, \quad \tfrac{3}{2}\mathbf{a}, \quad -\mathbf{b}, \quad \mathbf{a} + \mathbf{b}, \quad \mathbf{a} - \mathbf{b}.$$

2 Modelling displacements and velocities

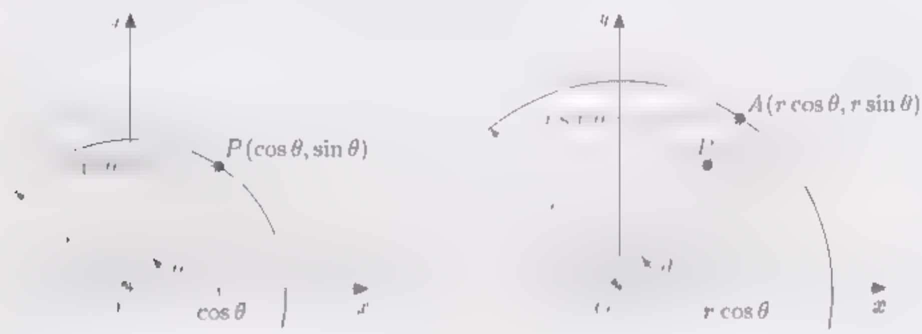
In this section you will see first how to convert vectors from geometric form, in terms of a magnitude and direction, to component form, and then how conversion in the opposite sense is accomplished. The ability to convert between these different forms of a vector is useful in certain problems involving displacement and velocity, as shown in Subsection 2.2.

2.1 From geometric to component form, and back

In some applications of vectors there is a need to move backwards and forwards between geometric form and component form; we deal here with how to achieve this.

To start with, we recall the definitions of cosine and sine that were given in Chapter A2. If P is a point on the unit circle, and the line segment OP makes an angle θ measured anticlockwise from the positive x-axis, then $\cos\theta$ is the x-coordinate of P and $\sin\theta$ is the y-coordinate of P (see Figure 2.1(a)). In other words, P has coordinates $(\cos\theta, \sin\theta)$.

See Chapter A2 Subsection 3.1.



(a) (b)

Figure 2.1 (a) Definition of $\cos\theta$ and $\sin\theta$ (b) Coordinates of A

Now suppose that the unit circle is scaled by a factor r in both the x- and y-directions, to produce a circle of radius r, as illustrated in Figure 2.1(b). If P is moved to the point A by this scaling, then the line segment OA has length r, and the coordinates of A are $(r\cos\theta, r\sin\theta)$.

The numbers r, θ are called the **polar coordinates** of A.

As stated in Subsection 1.2, the *coordinates* of the point A are the *components of the position vector* of A. Thus the vector $\overrightarrow{OA}$ has magnitude $|\overrightarrow{OA}| = r$ and component form $r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$.

Suppose that a vector $\mathbf{a}$ is given in terms of its magnitude $|\mathbf{a}|$ and direction θ. Then there is a unique point A in the plane such that the position vector of A is equal to $\mathbf{a}$, namely, the point which is distant $|\mathbf{a}|$ from O along a straight line that makes an angle θ with the positive x-axis (see Figure 2.2). But since $\mathbf{a} = \overrightarrow{OA}$ for this point A, the quantity $|\mathbf{a}|$ takes the place of r in the earlier discussion. We therefore have the following result.

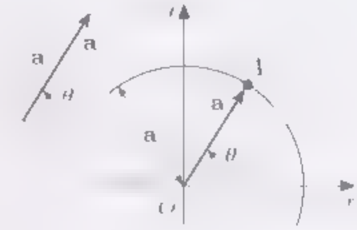


Figure 2.2 The vector $\mathbf{a}$ defines a point A via $\overrightarrow{OA} = \mathbf{a}$

From geometric to component form

Any vector $\mathbf{a}$, with direction θ, has component form

$$\mathbf{a} = |\mathbf{a}|\cos\theta\,\mathbf{i} + |\mathbf{a}|\sin\theta\,\mathbf{j}.$$

In other words, its $\mathbf{i}$-component is $a_1 = |\mathbf{a}|\cos\theta$ and its $\mathbf{j}$-component is $a_2 = |\mathbf{a}|\sin\theta$.

Example 2.1 Finding the component form

The vector $\mathbf{a}$ has magnitude 4 and direction 120°. Calculate the component form of $\mathbf{a}$, giving the components:

(a) correct to four decimal places; (b) as exact values.

Solution

(a) According to the formula in the box above, the component form is

$$\mathbf{a} = 4\cos(120^\circ)\mathbf{i} + 4\sin(120^\circ)\mathbf{j} = -2\mathbf{i} + 3.4641\mathbf{j} \quad \text{(to 4 d.p.)}.$$

A calculator was used to obtain the 4 d.p. result.

(b) Exact values for the components can be obtained where one of the 'special angles' 30°, 45° or 60° is seen to be involved. Here the given direction is 120°, which is 180° − 60°. Using the trigonometric identities

$$\cos(180^\circ - \theta) = -\cos\theta \quad \text{and} \quad \sin(180^\circ - \theta) = \sin\theta,$$

These identities were given in Chapter A2, Subsection 3.1, but for angles in radians rather than degrees.

we obtain

$$\begin{aligned}\mathbf{a} &= 4\cos(120^\circ)\mathbf{i} + 4\sin(120^\circ)\mathbf{j} \\ &= -4\cos(60^\circ)\mathbf{i} + 4\sin(60^\circ)\mathbf{j} \\ &\quad [\text{illegible}] \\ &= -2\mathbf{i} + 2\sqrt{3}\mathbf{j}.\end{aligned}$$

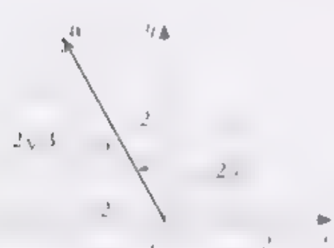


Figure 2.3 The vector of Example 2.1

The outcome is illustrated in Figure 2.3. A diagram such as this could have been used, instead of the trigonometric identities above, to decide upon the sign and size of each component.

Activity 2.1 Finding the component form

(a) For each case below, calculate the component form of the vector $\mathbf{a}$ whose magnitude $|\mathbf{a}|$ and direction θ are given, specifying the components correct to four decimal places.

(i) $|\mathbf{a}| = 3$, $\theta = 110^\circ$ (ii) $|\mathbf{a}| = 2.5$, $\theta = -20^\circ$

(b) For each case below, calculate the component form of the vector $\mathbf{a}$ whose magnitude $|\mathbf{a}|$ and direction θ are given, specifying the components as exact values.

(i) $|\mathbf{a}| = 1$, $\theta = 45^\circ$ (ii) $|\mathbf{a}| = 5$, $\theta = -150^\circ$

Solutions are given on page 59.

You have seen how any vector given in geometric form, in terms of magnitude and direction, can be written in component form. You will now see how conversion in the opposite sense may be achieved, starting from component form. In other words, given a vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, what are its magnitude $|\mathbf{a}|$ and direction θ?

The first part of this question was answered in Subsection 1.2: by Pythagoras' Theorem, the magnitude of a vector is given in terms of its components by

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}.$$

As regards finding the direction, we first deal with some special cases, which are illustrated in Figure 2.4.

Figure 2.4 Four special cases for the direction of a vector

These cases may be described as follows:

- if $\mathbf{a} = a_1\mathbf{i}$ (that is, if $a_2 = 0$), then $\theta = \begin{cases} 0, & \text{if } a_1 > 0, \\ 180°, & \text{if } a_1 < 0; \end{cases}$
- if $\mathbf{a} = a_2\mathbf{j}$ (that is, if $a_1 = 0$), then $\theta = \begin{cases} 90°, & \text{if } a_2 > 0, \\ -90°, & \text{if } a_2 < 0. \end{cases}$

Suppose now that none of these special cases applies. It follows from earlier in this subsection that the magnitude $|\mathbf{a}|$, direction θ and components a_1, a_2 are related by the pair of equations

$$|\mathbf{a}| \cos\theta = a_1 \quad \text{and} \quad |\mathbf{a}| \sin\theta = a_2.$$

Previously, these gave the components in terms of known magnitude and direction, but now the situation is reversed: a_1 and a_2 are known, and (with $|\mathbf{a}|$ already found as above) these equations must be *solved* for the remaining unknown, θ. The equations, taken together, have a unique solution for θ within the range $-180° < \theta \leq 180°$.

There are various ways of finding this solution. We shall use an approach that involves taking an inverse tangent. If the equation $|\mathbf{a}|\sin\theta = a_2$ is divided by the equation $|\mathbf{a}|\cos\theta = a_1$, then we obtain

$$\tan\theta = \frac{a_2}{a_1}.$$

Note that the possibility of division by zero is avoided here, because $a_1 = 0$ corresponds to one of the special cases dealt with above [illegible]

This suggests that the direction θ is found by taking the arctan of both sides of the last equation. However, care is needed at this point, because the equation for $\tan\theta$ has *more than one solution* within the desired range $-180° < \theta \leq 180°$. (For example, we have $\tan(45°) = 1$, but also $\tan(-135°) = 1$.) The following strategy shows how to choose between the possibilities on offer for θ.

The function arctan is the inverse function of the function $f(x) = \tan x$ with domain $(-90°, 90°)$; see Chapter A3, Subsection 4.2, in which radian measure was used. So values of arctan lie in the range from $-90°$ to $90°$. In particular, since $|a_2/a_1| > 0$, we have $0 < \arctan(|a_2/a_1|) < 90°$.

◇ Evaluate $\phi = \arctan(|a_2/a_1|)$. This gives the angle (between 0 and 90° since $|a_2/a_1| > 0$) that the position vector $\overrightarrow{OA} = \mathbf{a}$ makes with the positive or negative x-axis, where the point A has coordinates (a_1, a_2). There are then four possibilities, as detailed in Figure 2.5(a).

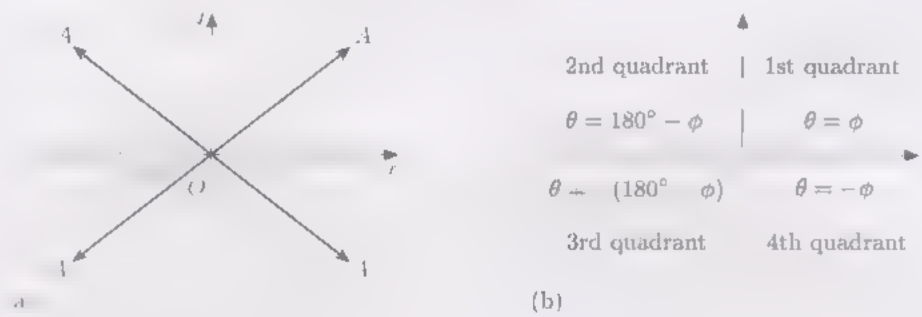


Figure 2.5 (a) The point A lies in one of the four quadrants (b) Corresponding value of θ

◇ The quadrant in which A lies is found from its coordinates, (a_1, a_2), and this determines which of the four possibilities applies. The corresponding value of θ is then found from that of ϕ as indicated in Figure 2.5(b).

The following box summarises how to find the geometric form of a vector from its component form.

From component to geometric form

A vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ has magnitude

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}.$$

If the vector is non-zero, then its direction θ is obtained as follows.

1. If $\mathbf{a} = a_1\mathbf{i}$ or $\mathbf{a} = a_2\mathbf{j}$, then θ can be found directly, as shown in Figure 2.4.
2. For a case other than those above, find $\phi = \arctan(|a_2/a_1|)$.
3. Find in which quadrant of the plane the point $A\,(a_1, a_2)$ lies. The value of θ (in terms of ϕ) is then given according to Figure 2.5(b).

Example 2.2 Finding the geometric form

Find the magnitude and direction of each of the following vectors.

(a) $\mathbf{a} = \sqrt{3}\mathbf{i} + \mathbf{j}$ (b) $\mathbf{b} = \mathbf{i} - \sqrt{3}\mathbf{j}$

(c) $\mathbf{c} = -\mathbf{i} - \sqrt{3}\mathbf{j}$ (d) $\mathbf{d} = -\sqrt{3}\mathbf{i} + \mathbf{j}$

Solution

The magnitude of all these vectors is the same, namely,

$$\sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2.$$

It remains to find the direction θ in each case. Arrows to represent the four vectors are shown in Figure 2.6.

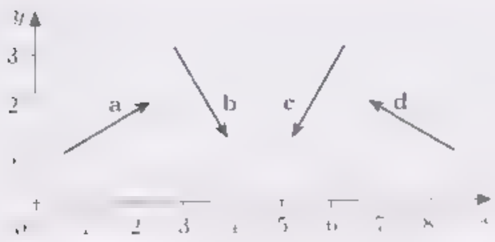


Figure 2.6 The four vectors

(a) For $\mathbf{a} = \sqrt{3}\mathbf{i} + \mathbf{j}$, we have $a_1 = \sqrt{3}$, $a_2 = 1$ and

$$\phi = \arctan(1/\sqrt{3}) = 30^\circ.$$

Also, $(\sqrt{3}, 1)$ lies in the first quadrant, so the direction of $\mathbf{a}$ is $\theta = \phi = 30^\circ$.

(b) For $\mathbf{b} = \mathbf{i} - \sqrt{3}\mathbf{j}$, we have $b_1 = 1$, $b_2 = -\sqrt{3}$ and

$$\phi = \arctan(\,|-\sqrt{3}\,|) = \arctan(\sqrt{3}\,) = 60^\circ.$$

Also, $(1, -\sqrt{3}\,)$ lies in the fourth quadrant, so the direction of $\mathbf{b}$ is $\theta = -\phi = -60^\circ$.

(c) For $\mathbf{c} = -\mathbf{i} - \sqrt{3}\mathbf{j}$, we have $c_1 = -1$, $c_2 = -\sqrt{3}$ and

$$\phi = \arctan(\,|-\sqrt{3}/(-1)|) = \arctan(\sqrt{3}\,) = 60^\circ.$$

Also, $(-1, -\sqrt{3}\,)$ lies in the third quadrant, so the direction of $\mathbf{c}$ is $\theta = -(180^\circ - \phi) = -120^\circ$.

(d) For $\mathbf{d} = -\sqrt{3}\mathbf{i} + \mathbf{j}$, we have $d_1 = -\sqrt{3}$, $d_2 = 1$ and

$$\phi = \arctan(\,|1/(-\sqrt{3}\,)|) = \arctan(1/\sqrt{3}\,) = 30^\circ.$$

Also, $(-\sqrt{3}, 1)$ lies in the second quadrant, so the direction of $\mathbf{d}$ is $\theta = 180^\circ - \phi = 150^\circ$.

Comment

All of these cases involved exact values for ϕ and θ. In general, this will not be so.

Activity 2.2 Finding the geometric form

Find the magnitude and direction of each of the following vectors. Give answers as exact values where possible; otherwise, give the direction correct to one decimal place. (There is no need to reduce magnitudes to decimal form.)

(a) $\mathbf{e} = -3\mathbf{i} + 3\mathbf{j}$ (b) $\mathbf{f} = 4\mathbf{i} - 2\mathbf{j}$

(c) $\mathbf{g} = -2\mathbf{i} - 3\mathbf{j}$ (d) $\mathbf{h} = -3.5\mathbf{j}$

Solutions are given on page 60.

The following activity illustrates how the conversion processes outlined above may come in useful. If two vectors are given in geometric form, and their sum is sought in the same form, one approach is to convert each of the vectors into component form, add their corresponding components, and then convert the sum back to geometric form.

Another approach is to apply the Triangle Rule, but as pointed out in Activity 1.7, estimates obtained from measurement will be imprecise.

Activity 2.3 Finding the sum in geometric form, via components

Find the magnitude and direction of the sum of the two vectors which were specified in Activity 2.1(a), giving your answers correct to two decimal places. (You will need to choose different labels for the two vectors.)

A solution is given on page 60.

2.2 Displacements and velocities

In this subsection we look at applications of the vector ideas introduced so far to *displacements* and *velocities*. The examples feature directions referred to points of the compass, known as *bearings*.

Bearings

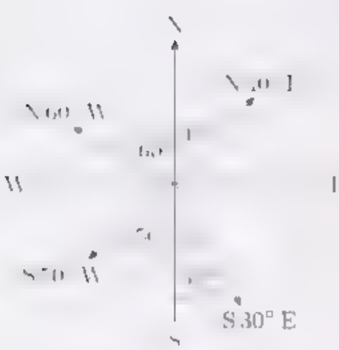


Figure 2.7 Examples of bearings

In Subsection 1.2, the direction of Leeds relative to Bristol was described as '15° to the East of due North', which was abbreviated to N 15°E. This is an instance of a **bearing**. Directions on the ground are typically given like this, in terms of the directions North (N), South (S), East (E) and West (W). For a direction other than N, S, E or W, the conventional notation for bearings involves starting from either North or South, and then specifying an angle (up to 90°) towards the East or West. The possibilities that arise are illustrated by specific examples in Figure 2.7.

Often N 45°E is replaced by NE (North-East), and similarly for N 45°W (North-West), S 45°E (South-East) and S 45°W (South-West).

Displacements and bearings

As you saw in Subsection 1.2, the displacement from a point P to a point Q is the change of position between the two points, as described by the displacement vector $\overrightarrow{PQ}$. If P and Q represent places on the ground, then it is natural to use a bearing to describe the direction of Q from P. It is straightforward to move between bearings and vector directions (relative to Cartesian axes), once the Cartesian unit vectors have been specified in terms of compass directions.

Activity 2.4 Converting to and from bearings

Suppose that $\mathbf{i}$ is 1 km East and $\mathbf{j}$ is 1 km North.

(a) Find the direction θ that corresponds to each of the following bearings.

(i) N 40°E (ii) S 50°W

(b) Find the bearing that corresponds to each of the following directions.

(i) $\theta = -70^\circ$ (ii) $\theta = 120^\circ$

Solutions are given on page 60.

The next activity asks you to apply to displacements the strategies from Subsection 2.1 for converting between geometric and component form.

Activity 2.5 Displacements in geometric and component form

Suppose that $\mathbf{i}$ is 1 km East and $\mathbf{j}$ is 1 km North.

(a) The displacement from Bristol to Leeds is given by the vector $\mathbf{d} = 77\mathbf{i} + 286\mathbf{j}$. Verify that $\mathbf{d}$ may also be described (as claimed in Subsection 1.2) by 296 km at N 15°E.

(b) The displacement from Exeter to Belfast is 465 km at N 21°W.

(i) How far East or West of Exeter is Belfast?

(ii) How far North or South of Exeter is Belfast?

(c) Specify the displacement vector from Belfast to Exeter:

(i) in component form;

(ii) in terms of direct distance and a bearing.

Solutions are given on page 61.

The approach outlined in Activity 2.3 can be applied to two or more displacements, in order to find their resultant. This is demonstrated in the following example.

Example 2.3 Finding the resultant displacement

A surveyor walks 200 metres due North. She then turns clockwise through 120° and walks 100 metres, after which she walks 300 metres due West. Find her resultant displacement, relative to her starting position, in terms of direct distance and a bearing. Give your answers correct to one decimal place.

Solution

Let $\mathbf{i}$ be 1 m East and let $\mathbf{j}$ be 1 m North. Suppose that the displacement vectors for the three phases of the walk are denoted by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. The path of the surveyor is sketched in Figure 2.8.

Note that, here and in subsequent vector diagrams, the Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$ are not drawn to the scale indicated for other vectors

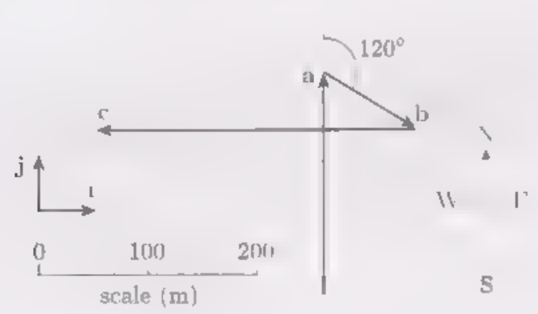


Figure 2.8 The surveyor's path

Since the magnitudes of $\mathbf{i}$ and $\mathbf{j}$ are each 1 m, the magnitudes of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are measured in metres. This convention is adopted in all similar problems in this chapter.

The vector $\mathbf{a}$ has magnitude $|\mathbf{a}| = 200$ and direction 90°. The vector $\mathbf{b}$ has magnitude $|\mathbf{b}| = 100$ and direction $90° - 120° = -30°$. The vector $\mathbf{c}$ has magnitude $|\mathbf{c}| = 300$ and direction 180°. Hence the respective component forms are

$$\mathbf{a} = 200\cos(90°)\mathbf{i} + 200\sin(90°)\mathbf{j} = 200\mathbf{j},$$
$$\mathbf{b} = 100\cos(-30°)\mathbf{i} + 100\sin(-30°)\mathbf{j} = 50\sqrt{3}\mathbf{i} - 50\mathbf{j},$$
$$\mathbf{c} = 300\cos(180°)\mathbf{i} + 300\sin(180°)\mathbf{j} = -300\mathbf{i}.$$

The resultant displacement is

Note the convenient extension of the use of the symbol $\simeq$ to vector contexts.

$$\mathbf{d} = \mathbf{a} + \mathbf{b} + \mathbf{c} = (50\sqrt{3} - 300)\mathbf{i} + (200 - 50)\mathbf{j} \simeq -213.3975\mathbf{i} + 150\mathbf{j}.$$

The strategy on page 22 can now be applied. The vector $\mathbf{d}$ has magnitude

$$|\mathbf{d}| \simeq \sqrt{(-213.3975)^2 + 150^2} \simeq 260.8.$$

Since the components of $\mathbf{d}$ are $d_1 \simeq -213.3975$, $d_2 = 150$, we have

$$\phi \simeq \arctan(|150/(-213.3975)|) = \arctan(150/213.3975) \simeq 35.1°.$$

Also, $(-213.3975, 150)$ lies in the second quadrant, so the direction of $\mathbf{d}$ is $\theta = 180° - \phi \simeq 144.9°$. This corresponds to the bearing N 54.9° W.

Thus the surveyor's resultant displacement, relative to her starting point, is 260.8 m at N 54.9° W. (An arrow representing the resultant vector $\mathbf{d}$ should have its tail at the tail of the arrow representing $\mathbf{a}$ and its tip at the tip of the arrow representing $\mathbf{c}$. You might like to check the calculation of $\mathbf{d}$ by drawing, to scale, a suitable arrow.)

Activity 2.6 Finding the resultant displacement

The displacement from Exeter to Belfast is 465 km at N 21° W (as given in Activity 2.5(b)). The displacement from Belfast to Edinburgh is 230 km at N 48° E. Find, in terms of direct distance and a bearing, the displacement from Exeter to Edinburgh. Give your answers to the nearest kilometre or degree.

A solution is given on page 61.

Velocity

Another vector quantity which crops up frequently in applied mathematics is *velocity*. In everyday English, the words 'speed' and 'velocity' mean much the same as each other, but in scientific parlance there is a significant difference between them.

Velocity and speed

- The **velocity** $\mathbf{v}$ of an object is its rate of change of position. This is a (vector) measure of how fast it is moving *and* of its direction of motion.
- The **speed** $|\mathbf{v}|$ of an object is a (scalar) measure of how fast it is moving, *irrespective* of its direction of motion. As the notation indicates, the speed is the magnitude of the velocity vector. The SI unit for speed is metres per second ($\mathrm{m\,s^{-1}}$).

Here we consider only constant velocities, which have both constant speed and fixed direction. The direction of a velocity can be expressed in several ways, but it is often given as a bearing, as for displacements. Indeed, constant velocities are closely related to displacements. If an object travels with constant velocity $\mathbf{v}\,\mathrm{m\,s^{-1}}$ (metres per second) for t seconds, then its displacement $\mathbf{d}$ metres over that time is given by $\mathbf{d} = t\mathbf{v}$. So each constant velocity is a scalar multiple of a displacement ($\mathbf{v} = (1/t)\mathbf{d}$), and vice versa.

Provided $t > 0$, the vectors $\mathbf{v}$ and $\mathbf{d}$ have the same direction.

One particular example is *wind velocity*. Note that when the direction of a wind is specified in English (as in 'a westerly wind'), this describes the direction from which the wind blows, and not the direction to which it moves

Activity 2.7 Velocity vector for wind

Let $\mathbf{i}$ be $1\,\mathrm{m\,s^{-1}}$ East and let $\mathbf{j}$ be $1\,\mathrm{m\,s^{-1}}$ North. Give in component form the velocity vector for a south-westerly wind of speed $2\,\mathrm{m\,s^{-1}}$.

A solution is given on page 61.

The sorts of problems that arise for velocities are similar to those that you have met already for displacements, as the following example illustrates.

Example 2.4 Crossing the river

A river flows due East at a speed of $0.3\,\mathrm{m\,s^{-1}}$. A boy in a rowing boat, who can row at $0.5\,\mathrm{m\,s^{-1}}$ in still water, starts from a point on the South bank and steers at right-angles to the bank. The boat is also blown by a wind with speed $0.4\,\mathrm{m\,s^{-1}}$ from a N 20°E direction.

(a) Find the resultant velocity of the boat, in terms of its speed (correct to two decimal places) and a bearing (to the nearest degree).

(b) Suppose that the river has constant width 10 metres. How long does it take the boy to cross the river, and how far upstream or downstream has he then travelled?

Solution

(a) Let $\mathbf{i}$ be $1\,\mathrm{m\,s^{-1}}$ East and let $\mathbf{j}$ be $1\,\mathrm{m\,s^{-1}}$ North. Assume that the boy rows throughout at the maximum speed of which he is capable. Suppose that $\mathbf{v}_b$ is the velocity of the boat in still water, $\mathbf{v}_r$ is the velocity of the river and $\mathbf{v}_w$ is the velocity of the wind. These vectors are shown in Figure 2.9.

From the information given, $\mathbf{v}_b$ has magnitude $|\mathbf{v}_b| = 0.5$ and direction $\theta_b = 90°$, while $\mathbf{v}_r$ has magnitude $|\mathbf{v}_r| = 0.3$ and direction $\theta_r = 0$.

The wind comes *from* N 20°E and hence blows *towards* S 20°W, for which the direction is $-(90° + 20°)$. Hence the vector $\mathbf{v}_w$ has magnitude $|\mathbf{v}_w| = 0.4$ and direction $\theta_w = -110°$.

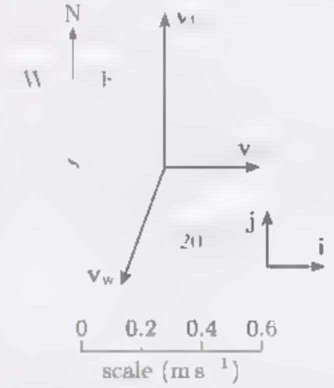


Figure 2.9 Three velocity vectors

The first two component forms can in this case be written down directly from Figure 2.9.

The component forms of the vectors are therefore

$$\mathbf{v}_b = 0.5\mathbf{j},$$
$$\mathbf{v}_r = 0.3\mathbf{i},$$
$$\mathbf{v}_w = 0.4\cos(-110^\circ)\mathbf{i} + 0.4\sin(-110^\circ)\mathbf{j} \simeq -0.1368\mathbf{i} - 0.3759\mathbf{j}$$

The resultant velocity is

$$\mathbf{v} = \mathbf{v}_b + \mathbf{v}_r + \mathbf{v}_w \simeq (0.3 - 0.1368)\mathbf{i} + (0.5 - 0.3759)\mathbf{j}$$
$$\simeq 0.1632\mathbf{i} + 0.1241\mathbf{j}.$$

The strategy on page 22 can now be applied. The resultant speed of the boat is

$$|\mathbf{v}| \simeq \sqrt{(0.1632)^2 + (0.1241)^2} \simeq 0.21.$$

Since the components of $\mathbf{v}$ are $v_1 \simeq 0.1632$, $v_2 \simeq 0.1241$, we have

$$\phi \simeq \arctan(0.1241/0.1632) \simeq 37^\circ.$$

Also, $(0.1632, 0.1241)$ lies in the first quadrant, so the direction of $\mathbf{v}$ is $\theta = \phi \simeq 37^\circ$. This corresponds to the bearing N 53°E.

Thus the resultant velocity is $0.21\,\mathrm{m\,s^{-1}}$ at N 53°E.

(b) The $\mathbf{i}$-component of $\mathbf{v}$ indicates how fast the boat travels downstream (or upstream, if $v_1 < 0$), while the $\mathbf{j}$-component, v_2, is the rate of progress across the river. Since $v_2 \simeq 0.1241$ (in $\mathrm{m\,s^{-1}}$), the 10-metre width of the river will be crossed in $10/0.1241 \simeq 81$ seconds.

Here we ignore the length of the boat!

In that time, the boat will have travelled $81 \times 0.1632 \simeq 13$ metres downstream.

Comment

It is natural to draw the arrows for velocity vectors in Figure 2.9 with their tails at the same point, since the boat can be thought of as moving under the simultaneous effect of the three velocities. When adding displacements, on the other hand, it is natural to think of the vectors taking effect consecutively and hence to place the arrows nose to tail. This is simply a choice of visualisation, and does not affect the mathematics involved. However, placing the arrows nose to tail does give some indication of the magnitude and direction of the resultant vector, which provides a rough check on the calculation.

Activity 2.8 Ship in a current

A ship has a speed in still water of $5\,\mathrm{m\,s^{-1}}$ and is pointed in the direction S 50°W, but there is a current of speed $2\,\mathrm{m\,s^{-1}}$ flowing towards the direction N 40°W. Find the resultant velocity of the ship, in terms of its speed (correct to one decimal place) and a bearing (to the nearest degree).

A solution is given on page 62.

Summary of Section 2

This section has introduced:

- conversion of a vector from geometric form (in terms of magnitude and direction) to component form;
- conversion of a vector from component form to geometric form;
- bearings;
- velocity and speed;
- finding resultant displacements and velocities in geometric form, via the use of components.

Exercises for Section 2

Exercise 2.1

A vector $\mathbf{a}$ has magnitude $|\mathbf{a}| = 7$ and direction $\theta = -70°$. Calculate the component form of $\mathbf{a}$, giving the components correct to two decimal places.

Exercise 2.2

Find the magnitude and direction of each of the vectors $\mathbf{a} = -3\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = 6\mathbf{i} - \mathbf{j}$ and of their sum $\mathbf{a} + \mathbf{b}$, giving your answers correct to one decimal place.

Exercise 2.3

The displacement from Derby to Birmingham is 57 km at S 30° W. The displacement from Derby to Leicester is 32 km at S 45° E. In terms of direct distance and a bearing, find:

(a) the displacement from Leicester to Derby;

(b) the displacement from Leicester to Birmingham, giving your answers correct to one decimal place.

Exercise 2.4

An aeroplane has a speed in still air of $50\,\mathrm{m\,s^{-1}}$ and is pointed in the direction N 20° E, but it flies in a wind of speed $10\,\mathrm{m\,s^{-1}}$ blowing from N 70° W. Find the velocity of the aeroplane relative to the ground, in terms of its speed and a bearing. Give your answers correct to one decimal place.

3 Sine Rule and Cosine Rule

In Section 2 you saw how conversion to components could be applied to two vectors, given in geometric form, in order to calculate their resultant, also eventually in geometric form. In this section you will see an alternative approach to such problems, which does not involve components.

This approach builds on the Triangle Rule for adding two vectors in geometric form. It depends upon being able to solve triangles, that is, to find all of the side lengths and angles of a triangle, given only partial information about these. You saw how to carry out this process for right-angled triangles in Chapter A2. The process is now extended to triangles in general, by means of the *Sine Rule* and *Cosine Rule*.

These rules are important trigonometric results in their own right. Their application to vector examples comes in Subsection 3.3, but before that you will find that vectors do not appear at all.

3.1 Sine Rule

The Sine Rule was introduced in Chapter A2, Subsection 3.2, in a non-assessed part of the text. It is reintroduced here as if you have not seen it before.

In this section we need a labelling system for the side lengths and angles of a general triangle. A standard convention for such a labelling is shown in Figure 3.1(a). As you can see, the vertex labels A, B and C are also used to denote the corresponding angle sizes, while the side lengths opposite these angles are denoted respectively by a, b and c.

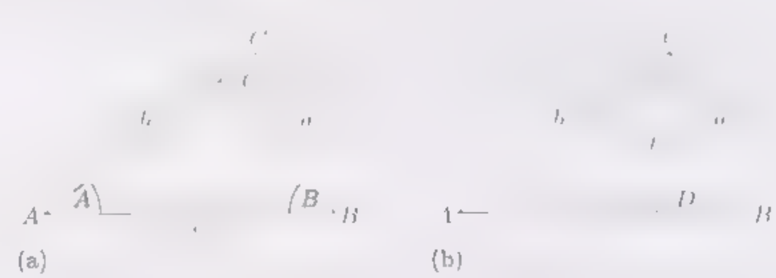


Figure 3.1 A labelling convention

The rules for solving general triangles depend essentially upon being able to view them as made up of right-angled triangles. For example, by dropping the perpendicular from C to AB in $\triangle ABC$, we divide this triangle into two smaller ones with right angles, $\triangle ADC$ and $\triangle CDB$, as shown in Figure 3.1(b). Denoting the length of CD by h, we have $h = b\sin A$ (in $\triangle ADC$) and $h = a\sin B$ (in $\triangle CDB$). Since

$$h = b\sin A = a\sin B,$$

a, b, c are non-zero since they are the side lengths of a triangle

it follows on dividing through by ab that

$$\frac{\sin A}{a} = \frac{\sin B}{b}.$$

If instead the perpendicular from A to BC is dropped, then we find similarly that

$$\frac{\sin B}{b} = \frac{\sin C}{c}.$$

This may be combined with the previous result in the form

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

or, on taking the reciprocal of each fraction,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Either of these last two formulas is a statement of the *Sine Rule*.

> **Sine Rule**
>
> For any triangle, the side lengths a, b, c and corresponding opposite angles A, B, C are related by the formulas
>
> $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$
>
> and
>
> $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Note that if $A = 90°$, then the Sine Rule gives

$$\sin B = \frac{b}{a}, \quad \sin C = \frac{c}{a}.$$

In each case, this is just the familiar ratio

$$\frac{\text{opposite}}{\text{hypotenuse}}$$

for the sine of an angle in a right-angled triangle.

Given the length of one side for any triangle whose angles are known, this rule permits us to deduce the lengths of the remaining sides. Since the angles of a triangle sum to 180°, it suffices to be told at the start the sizes of any two of the angles, from which the third can always be deduced.

Example 3.1 Applying the Sine Rule

Two angles in the triangle ABC are $A = 73°$ and $B = 49°$, while one side length is $a = 12.2$ (see Figure 3.2). Find the other two side lengths, b and c, giving your answers correct to one decimal place.

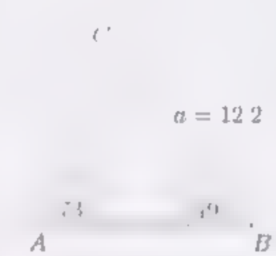


Figure 3.2 Triangle ABC

You may find it helpful to draw a figure for a geometric problem, if one is not given.

Solution

Using the Sine Rule in the form

$$\frac{a}{\sin A} = \frac{b}{\sin B},$$

we obtain

$$b = \frac{a \sin B}{\sin A} = \frac{12.2 \sin(49°)}{\sin(73°)} = 9.6 \quad \text{(to 1 d.p.)}.$$

To calculate c, we first need to find the angle C, which is given by

$$C = 180° - 73° - 49° = 58°.$$

Now we use the Sine Rule once again, this time in the form

$$\frac{a}{\sin A} = \frac{c}{\sin C},$$

to obtain

$$c = \frac{a \sin C}{\sin A} = \frac{12.2 \sin(58°)}{\sin(73°)} = 10.8 \quad \text{(to 1 d.p.)}.$$

Hence $b = 9.6$ and $c = 10.8$ to one decimal place.

In this example, the unit of length was omitted since it plays no role in the calculation. In all similar problems, the same convention is adopted.

Activity 3.1 Applying the Sine Rule

Two angles in the triangle ABC are $A = 87°$ and $B = 63°$, while one side length is $c = 3.2$. Find the side length a, giving your answer correct to one decimal place.

A solution is given on page 62.

The Sine Rule can also be applied in some cases to find an unknown angle in a triangle for which only one of the angles but two sides are given. However, this may not be straightforward. For example, consider a triangle ABC in which $a = 5$, $b = 3$ and $B = 30°$. We may deduce from the Sine Rule that

$$\sin A = \frac{a \sin B}{b} = \frac{5 \times \frac{1}{2}}{3} = \frac{5}{6}.$$

The function arcsin is the inverse function of the function $f(x) = \sin x$ with domain $[-90°, 90°]$; see Chapter A3, Subsection 4.2, in which radian measure was used.

So the values of arcsin lie in the range from $-90°$ to $90°$.

In particular, if $0 < y < 1$, then $0 < \arcsin y < 90°$, and $180° - \arcsin y$ is obtuse.

However, since $\sin(180° - \theta) = \sin\theta$, we cannot obtain from this a unique value for A (in the range $0 < A < 180°$). In fact, we have

$$A = \arcsin \tfrac{5}{6} \simeq 56.44° \quad \text{or} \quad A \simeq 180° - 56.44° = 123.56°.$$

These possible outcomes correspond to the feasible vertex positions A_1 and A_2 shown in Figure 3.3(a). More information (for example, the length of the third side of the triangle) is needed in order to decide between the two possibilities.

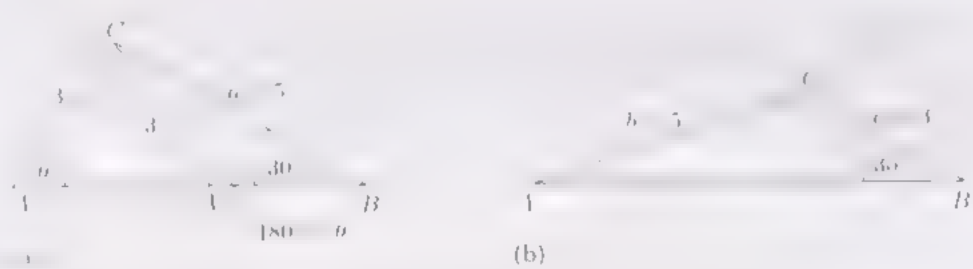


Figure 3.3 Triangles with a, b and B given: (a) $a > b$; (b) $a < b$

No other line segment of length 5 can be drawn from C and just reach the side opposite angle C.

As a further example, suppose that $a = 3$, $b = 5$ and $B = 30°$, as shown in Figure 3.3(b). There is now no ambiguity about how the triangle may be drawn, but the equation for A obtained from the Sine Rule, $\sin A = \frac{3}{10}$, still has two solutions between 0 and 180°. So one of the solutions of $\sin A = \frac{3}{10}$ cannot correspond to a real triangle. In general, how can we choose between two potential values of an angle in such a case? The following intuitive result, which can be proved by using the Sine Rule, provides the answer.

In other words, larger angles (within a triangle) are found opposite longer sides, and vice versa

> In a triangle ABC,
>
> if $a < b$ then $A < B$.
>
> and vice versa.

In the case of the triangle in Figure 3.3(b), we have $a = 3$ and $b = 5$, so $a < b$ and hence $A < B = 30°$. Thus $A = \arcsin \frac{3}{10} \simeq 17.46°$ is the required angle. (The other solution of the equation $\sin A = \frac{3}{10}$ is $180° - 17.46° = 162.54°$, which does not correspond to a real triangle.)

Note that the boxed result is satisfied for both triangles in Figure 3.3(a). Here $a > b$ and both $A = 56.44°$ and $A = 123.56°$ are greater than $B = 30°$.

Example 3.2 Applying the Sine Rule again

One angle in the triangle ABC is $A = 37°$, and two side lengths are $a = 4.6$ and $c = 2.1$. Find the angle C.

Solution

Using the Sine Rule in the form

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

we obtain

$$\sin C = \frac{c \sin A}{a} = \frac{2.1 \sin 37°}{4.6} \simeq 0.2747.$$

This gives the possible solutions

$$C \simeq \arcsin(0.2747) \simeq 15.94° \quad \text{and} \quad C \simeq 180° - 15.94° = 164.06°.$$

However, since $c < a$, we can deduce that $C < A = 37°$, which rules out the second possible solution. Hence we have $C \simeq 15.94°$.

Comment

There are other ways of showing that the 'solution' $C \simeq 164.06°$ is not possible within the triangle. For example, adding this to $A = 37°$ gives a value for $A + C$ that is greater than 180°, in contradiction of the angle sum in any triangle.

Activity 3.2 Applying the Sine Rule again

One angle in the triangle ABC is $B = 35°$, and two side lengths are $a = 2.7$ and $b = 5.1$. Find the angle A, giving your answer correct to one decimal place.

A solution is given on page 62.

3.2 Cosine Rule

The Sine Rule cannot be used directly when we are given the lengths of two sides of a triangle and the angle between them, even though the specification of these quantities is sufficient to define the triangle configuration completely. The rule developed in this subsection allows us to solve triangles from such given information, by determining the third side length. Alternatively, it can be used to find any angle of a triangle for which only the three side lengths are known at the outset.

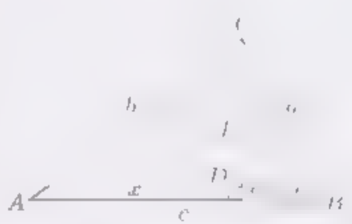


Figure 3.4 Dropping a perpendicular

We start once again with the general triangle ABC of Figure 3.1(a), and drop a perpendicular from C onto AB, to meet AB at D (see Figure 3.4). As before, the length of CD is denoted by h. Also, the length of AD is denoted by x, so the length of BD is $c - x$.

From considering $\triangle ADC$, we have

$$h = b \sin A \quad \text{and} \quad x = b \cos A.$$

This proof (and that earlier for the Sine Rule) assumes that the perpendicular from C to AB meets AB inside the triangle. If it does not, as in Figure 3.5, then minor alterations to the proofs are required, but the results still hold as stated.

Application of Pythagoras' Theorem to $\triangle BDC$ then gives

$$\begin{aligned} a^2 &= (c-x)^2 + h^2 \\ &= (c - b\cos A)^2 + (b\sin A)^2 \\ &= c^2 - 2cb\cos A + b^2\cos^2 A + b^2\sin^2 A \\ &= c^2 - 2cb\cos A + b^2(\cos^2 A + \sin^2 A) \\ &= c^2 - 2cb\cos A + b^2 \quad (\text{since } \cos^2 A + \sin^2 A = 1). \end{aligned}$$

Hence we have

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

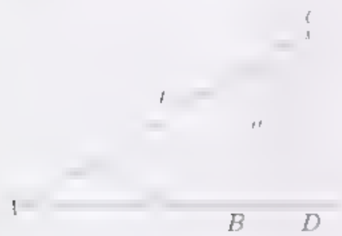


Figure 3.5 Perpendicular from C meets AB outside triangle ABC

This is known as the *Cosine Rule*. It can also be expressed as

$$b^2 = c^2 + a^2 - 2ca\cos B \quad \text{or} \quad c^2 = a^2 + b^2 - 2ab\cos C.$$

Alternatively, the cosine can be made the subject of the formula, giving the expressions

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca}, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Cosine Rule

For any triangle, the side lengths a, b, c and corresponding opposite angles A, B, C are related by the formulas

$$a^2 = b^2 + c^2 - 2bc\cos A, \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc},$$
$$b^2 = c^2 + a^2 - 2ca\cos B, \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca},$$
$$c^2 = a^2 + b^2 - 2ab\cos C, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Note that the Cosine Rule is a generalisation of Pythagoras' Theorem, to which it reduces if the value 90° is chosen for the angle in the formula.

The Cosine Rule can be applied where one knows either the three side lengths of a triangle, or two sides and the included angle.

Example 3.3 Applying the Cosine Rule

The triangle ABC has side lengths $a = 8$, $b = 10$ and $c = 15$. Find the largest angle of the triangle.

Solution

The largest angle is opposite the longest side, so it is the angle C which is required. Applying the Cosine Rule in the form

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

we obtain

$$\cos C = \frac{8^2 + 10^2 - 15^2}{2 \times 8 \times 10} = -0.381\,25,$$

so

$$C = \arccos(-0.381\,25) = 112.4^\circ \quad (\text{to 1 d.p.}).$$

Hence the largest angle of the triangle is 112.4° to one decimal place.

The function arccos is the inverse function of the function $f(x) = \cos x$ with domain $[0, 180°]$; see Chapter A3, Subsection 4.2, in which radian measure was used. So the values of arccos lie in the range from 0 to 180°, which is the range in which the angles of a triangle lie.

Comment

Note that there is no ambiguity about the answer here (unlike the earlier case, involving the use of arcsin) because arccos gives the unique solution between 0 and 180° to the equation $\cos C = -0.381\,25$.

Activity 3.3 Applying the Cosine Rule

The triangle ABC has side lengths $a = 10$, $b - 12$ and $c = 9$. Find the largest angle of the triangle, giving your answer correct to one decimal place.

A solution is given on page 62.

Having started the process of solving a triangle by applying the Cosine Rule, it is often simplest to achieve the next step by using the Sine Rule, as demonstrated in the following example.

Example 3.4 Solving a triangle

Solve the triangle ABC for which $a = 15.73$, $B = 121°$ and $c = 23.15$, giving your answers correct to two decimal places.

Solution

We are given two sides and the included angle, so the first step is to apply the Cosine Rule in the form

$$b^2 = c^2 + a^2 - 2ca\cos B.$$

This gives

$$b^2 = 23.15^2 + 15.73^2 - 2 \times 23.15 \times 15.73\cos(121°) \simeq 1158.457,$$

so $b = 34.04$ (to 2 d.p.).

Now that the lengths of the three sides are known, we need to find the remaining angles. For the first of these we could use the Cosine Rule again. However, it is less complicated to apply the Sine Rule, in the form

$$\frac{\sin C}{c} = \frac{\sin B}{b}.$$

This leads to

$$\sin C = \frac{c\sin B}{b} \simeq \frac{23.15\sin(121°)}{34.04} \simeq 0.5830.$$

Hence, since $c < b$ and so $C < B$, we have

$$C = 35.66° \quad \text{(to 2 d.p.)}.$$

The final angle is found by using the angle sum formula $A + B + C = 180°$. Thus

$$A = 180° - 121° - 35.66° = 23.34° \quad \text{(to 2 d.p.)}.$$

Hence, in $\triangle ABC$, we have $a = 15.73$, $b = 34.04$, $c = 23.15$, $A = 23.34°$, $B = 121°$ and $C = 35.66°$, all to two decimal places.

As usual in successive calculations of this type, it is good practice to retain the full accuracy provided by your calculator for intermediate values (e.g. for b here), even though values are not written down to that accuracy.

Activity 3.4 Solving a triangle

Solve the triangle ABC for which $A = 60°$, $b = 8$ and $c = 15$, giving your answers correct to one decimal place.

A solution is given on page 62.

3.3 Applications to vector examples

We have shown that the Sine Rule and Cosine Rule can be applied to solve triangles, which is precisely the process that is called for when using the Triangle Rule directly to obtain the resultant of two vectors in geometric form. All that is needed is to make the necessary connections between vector magnitudes and directions, on the one hand, and the side lengths and angles of a corresponding triangle, on the other. This is illustrated in the following example.

Example 3.5 Solving a triangle to find a resultant vector

A ship has a speed in still water of $10\,\mathrm{m\,s^{-1}}$ and is pointed in the direction N 30°E, but there is a current of speed $2\,\mathrm{m\,s^{-1}}$ flowing towards the direction SE. By applying the Cosine or Sine Rule, find the resultant velocity of the ship, in terms of its speed and a bearing. Give your answers correct to one decimal place.

Solution

Figure 3.6(a) shows the velocity vectors $\mathbf{v}_s$ for the ship in still water, $\mathbf{v}_c$ for the current and $\mathbf{v}$ for their resultant, according to the Triangle Rule for vector addition. The triangle in Figure 3.6(b) is congruent to that in Figure 3.6(a), but labelled according to the standard convention.

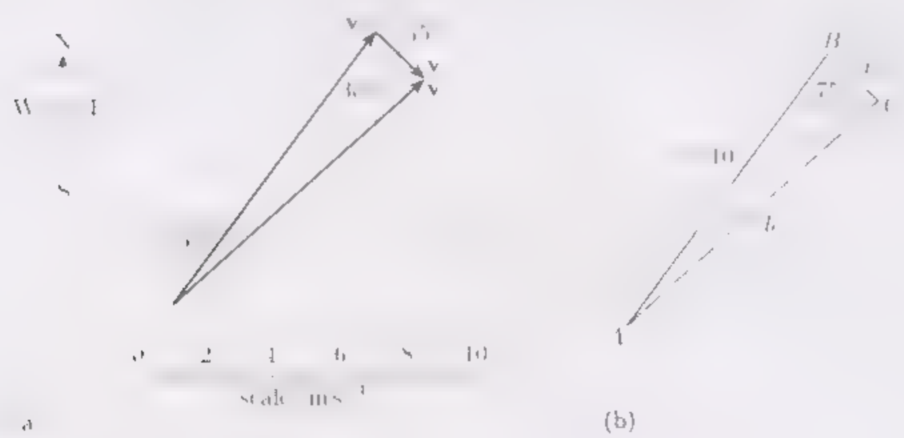


Figure 3.6 Triangle of velocities

Abstracting the vector information to the triangle ABC, we have $a = 2$, $c = 10$ and $B = 30° + 45° = 75°$. We wish to find b (the magnitude of $\mathbf{v}$) and A (since the bearing for $\mathbf{v}$ is N $(30° + A)$ E).

Using the Cosine Rule in the form

$$b^2 = c^2 + a^2 - 2ca\cos B,$$

we obtain

$$b^2 = 10^2 + 2^2 - 2 \times 10 \times 2\cos(75°) \simeq 93.6472,$$

so $b = 9.7$ (to 1 d.p.).

To find the angle A we now apply the Sine Rule, in the form

$$\frac{\sin A}{a} = \frac{\sin B}{b}.$$

This gives

$$\sin A = \frac{a \sin B}{b} \simeq \frac{2\sin(75°)}{9.7} \simeq 0.1996.$$

Since $a < b$, and hence $A < B$, we find $A \simeq 11.5°$.

Hence the resultant velocity of the ship is $9.7\,\mathrm{m\,s^{-1}}$ at N 41.5° E, to one decimal place.

Activity 3.5 Solving a triangle to find a resultant vector

By applying the Cosine or Sine Rule, repeat the task of Activity 2.6 on page 26.

A solution is given on page 63.

On some occasions, it may happen that the Triangle Rule involves the addition of two vectors which are at right angles. This simplifies matters considerably, as the next activity shows.

Activity 3.6 Vector arrows which form a right-angled triangle

Without using components, repeat the task of Activity 2.8 on page 28.

A solution is given on page 63.

When the resultant of two displacements or velocities in geometric form is sought, either the component approach from Subsection 2.2 or the 'solving triangle' approach in this subsection is appropriate. If more than two displacements or velocities are involved (as in Examples 2.3 and 2.4), then the component approach will usually be preferable, since the alternative requires the solution of more than one triangle.

Summary of Section 3

This section has introduced:

- the Sine Rule;
- the Cosine Rule;
- application of these rules to finding the resultant of displacements or velocities.

Exercises for Section 3

Exercise 3.1

Solve the triangle ABC for which $a = 12$, $B = 59^\circ$ and $C = 73^\circ$, giving the side lengths correct to one decimal place.

Exercise 3.2

The triangle ABC has side lengths $a = 4$, $b = 5$ and $c = 6$. Find the largest and smallest angles of this triangle, to the maximum accuracy which your calculator provides, and compare the values of the angles obtained.

Exercise 3.3

Without using components, repeat the task of Exercise 2.4 on page 29.

Exercise 3.4

An aeroplane has a speed in still air of $80\,\mathrm{m\,s^{-1}}$ and is pointed due North, but it flies in a wind of speed $40\,\mathrm{m\,s^{-1}}$ blowing from the South-West. By applying the Cosine or Sine Rule, find the velocity of the aeroplane relative to the ground, in terms of its speed and a bearing. Give your answers correct to one decimal place.

4 Modelling forces

How strong would a tow rope need to be to pull a car up a 1-in-6 hill? Would it be safe to put a grand piano in the middle of an upstairs room and, if so, what kind of block and tackle would be needed to lift it there? How do engineers design bridges and cranes that are strong enough not to collapse in use?

Questions like these belong to an area of applied mathematics called *mechanics*, which is concerned with the motion of objects and what causes or prevents it. We are mainly concerned here with the subject of *statics*, in which objects remain at rest under the effect of two or more *forces*, which is an important aim in the design of many engineering structures. Subsection 4.1 introduces forces, while Subsection 4.2 explains the mathematical condition for a balance of forces, which is sufficient to prevent an object being set in motion.

4.1 Introducing forces

If an object which is initially at rest starts to move, then it is being acted upon by a net *force*. Examples of forces occur all around us. The activities of pushing and pulling both involve the exertion of a force on the object which is pushed or pulled. Your own experience of pushing and pulling should give you an intuitive idea of the characteristics of forces.

[illegible] types of force.

To specify a force precisely, and describe its effect, we need to know the following.

(i) the object acted upon (a force has different effects on different objects – for example, if you push a supermarket trolley from rest, then the acceleration is greater for an empty trolley than for a full one, assuming that the same effort is expended in each case);

(ii) the point of application (the effect of a force depends upon where on the object it is applied – if you push near the side of a supermarket trolley, then you will produce a motion different to that obtained using the middle of the handle);

[illegible] region of contact.

(iii) the direction (forces in different directions have different effects – pushing the trolley's handle will cause forward motion, whereas pulling will move the trolley backwards);

(iv) the magnitude or strength (some forces are stronger than others – a harder push on the trolley will make it accelerate more rapidly).

Characteristics (iii) and (iv) suggest that it may be possible to model forces by vectors, since forces have both magnitude and direction. In fact, this turns out to be the case, because it can be shown experimentally that the way two forces combine is accurately represented by the Triangle Rule for vector addition. There is therefore good evidence to support the modelling assumption that *forces are represented by vectors*.

However, characteristics (i) and (ii) make clear that there is more to forces than their representation by vectors. In order to reduce the complexity as far as possible, we shall model any object to be considered here by a *particle*, which has only one point at which a force may be applied. This takes characteristic (ii) out of consideration. It also means that characteristic (i) is simplified, in that the only remaining distinguishing feature of an object, when modelled by a particle, is its *mass*.

The particle model

The **mass** of an object is a measure of the amount of matter that the object contains. The SI unit of mass is the **kilogram** (kg).

A **particle** is a material object whose size and internal structure may be neglected. It has mass but no size, and so occupies a single point in space.

Whether or not this model is a reasonable one in practice depends upon the context. For example, the Earth may reasonably be considered as a particle in an analysis of its orbital motion around the Sun, whereas the effects of spin on the trajectory of a tennis ball are neglected by the particle model. We shall consider here only circumstances in which it is reasonable to model the object under study by a particle.

The newton is named after Sir Isaac Newton (1642–1727).

Note that the mass of an object, as well as being the amount of matter that the object contains, is a measure of its reluctance to be accelerated by a given force.

In order to compare the sizes of forces, we need a unit of measurement. The SI unit of force is the **newton** (N). This is defined to be the magnitude of force which, when acting on an object of mass 1 kg, in the absence of other forces, causes the object to accelerate at 1 metre per second per second ($1\,\mathrm{m\,s^{-2}}$) in the direction of the force. Note that the **acceleration** of an object is its rate of change of velocity; it is a vector quantity.

Weight

The fact that, in the absence of other forces, all objects fall under gravitational attraction at the same rate was first realised by Galileo Galilei (1564–1642). He is reputed to have confirmed it by dropping two balls of different mass from the top of the Leaning Tower of Pisa, and noting that they reached the ground at the same instant.

The value of g varies slightly from place to place on the Earth's surface. To two decimal places, it is $9.78\,\mathrm{m\,s^{-2}}$ at the Equator and $9.83\,\mathrm{m\,s^{-2}}$ at the North Pole.

If an object is dropped, above but not far from the surface of the Earth, then it will start to fall, and the magnitude of the acceleration with which it falls is denoted by g. This *acceleration due to gravity* is the same for all objects, and has magnitude $g = 9.8\,\mathrm{m\,s^{-2}}$ to one decimal place. The *force due to gravity* on the object, which causes this acceleration, is also called the **weight** of the object. It follows from the definition of the newton, above, that an object of mass m kg has weight of magnitude mg N. Thus, for example, 1 newton is the weight magnitude of an average-sized tomato.

It is worth emphasising the difference between 'mass' and 'weight', as defined here, since in everyday speech these two words are sometimes used interchangeably. If you are told that something 'weighs 1 kilogram', this is really a concise way of saying that the weight magnitude close to the Earth's surface is that of a 1 kg mass. (Elsewhere, the weight magnitude may be different. For example, on the surface of the Moon, the weight of an object has magnitude about one sixth of that on Earth.) Note also that mass is a *scalar* quantity, whereas weight (along with other forces) is a *vector*. The direction of weight is vertically downwards, and we shall denote it by $\mathbf{W}$.

Thus far we have talked of forces causing acceleration, and if only one force acts, then accelerated motion occurs. What is more, the force due to gravity acts on all objects. It follows that if an object does *not* move, then *there must be at least two forces acting on it.* This will be the situation considered throughout the rest of this chapter.

If you hold a heavy object at rest in your hand, then you can feel the gravitational pull on it. This is because your muscles are providing the countervailing force which balances that of gravity. The combined effect of gravity and of your muscular exertions is that no motion occurs. We now consider two other ways by which gravity can be opposed and hence motion prevented.

Tension

Consider the situation shown in Figure 4.1(a), in which an object of mass 1 kg is suspended from a string whose top end is tied to a fixed point.

The object remains at rest, even though it is acted upon by its weight $\mathbf{W}$, directed vertically downwards. The string is therefore providing a balancing force, which must be in the opposite direction, vertically upwards. This force is called the **tension**, denoted by $\mathbf{T}$. Arrows to represent the two force vectors, $\mathbf{W}$ and $\mathbf{T}$, are shown in Figure 4.1(b), which is an example of a *force diagram* for the forces acting on the object.

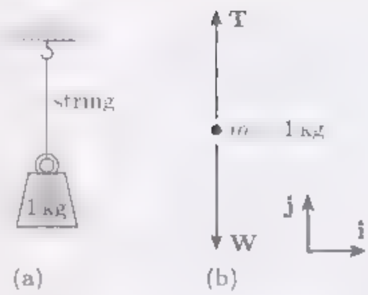


Figure 4.1 (a) A suspended object at rest
(b) Corresponding force diagram

The particle that models the object is represented by a circular dot, •, on the force diagram.

The magnitude of the tension $\mathbf{T}$ must be the same as that of $\mathbf{W}$, for if $|\mathbf{W}|$ were greater than $|\mathbf{T}|$ then the object would experience a net downward force and start to fall, whereas if $|\mathbf{W}|$ were less than $|\mathbf{T}|$ then the object would experience a net upward force and start to rise. We conclude that $\mathbf{T} = -\mathbf{W}$, which is equivalent to $\mathbf{T} + \mathbf{W} = \mathbf{0}$.

If the Cartesian unit vector $\mathbf{j}$ is chosen to represent 1 newton vertically upwards, as shown in Figure 4.1(b), then each of the vectors $\mathbf{W}$ and $\mathbf{T}$ can be written in component form. The mass of the object is $m = 1\,\text{kg}$. Taking $g = 9.8\,\text{m}\,\text{s}^{-2}$, we have that the weight $\mathbf{W}$ has magnitude $|\mathbf{W}| = mg = 9.8\,\text{N}$ and the same direction as $-\mathbf{j}$, so $\mathbf{W} = -9.8\mathbf{j}$. It follows that $\mathbf{T} = -\mathbf{W} = 9.8\mathbf{j}$

The tension has the property that it adjusts its magnitude so as to balance the weight of whatever object is tied to the bottom of the string, though there will be some upper limit to the force magnitude it can provide, beyond which the string breaks.

Normal reaction

Suppose now that the object of mass 1 kg is untied from the string and placed on a horizontal table top, as shown in Figure 4.2(a).

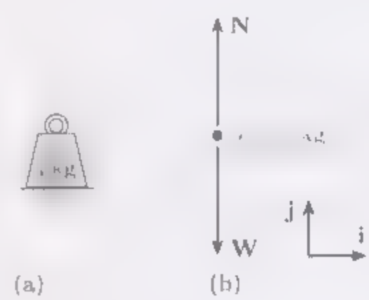


Figure 4.2 (a) An object at rest on a table top
(b) Corresponding force diagram

As before, the object remains at rest, and its weight $\mathbf{W}$ acts vertically downwards. There is now no pulling force to prevent its descent, as was provided previously by the tension in the string, but there is instead an upward pushing force provided by the table. This is called the **normal reaction** of the table on the object, denoted by $\mathbf{N}$. (In this context, *normal* means 'at right angles'.) Arrows to represent the two force vectors in this case, $\mathbf{W}$ and $\mathbf{N}$, are shown in the force diagram of Figure 4.2(b).

The mathematical details for $\mathbf{N}$ are very similar to those presented previously for $\mathbf{T}$, since $\mathbf{N}$ must balance $\mathbf{W}$ exactly to prevent motion. Hence we have $\mathbf{N} = -\mathbf{W}$ or, equivalently, $\mathbf{N} + \mathbf{W} = \mathbf{0}$. Once again, for the 1 kg mass, the component forms of the forces are $\mathbf{W} = -9.8\mathbf{j}$ and $\mathbf{N} = -\mathbf{W} = 9.8\mathbf{j}$.

Like the tension, the normal reaction adjusts its magnitude so as to balance the weight of whatever object is placed on the table, unless the object concerned is so massive that the table cannot support it and collapses.

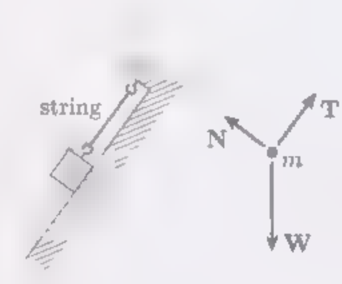


Figure 4.3 (a) An object at rest on a slope
(b) Corresponding force diagram

In the examples above, both of the tension and normal reaction acted vertically upwards, as they had to do in order to balance the weight acting vertically downwards. However, tension and normal reaction need not in general act vertically, where more than two forces are involved. In fact, the tension force will always pull along the line of the string, whereas the normal reaction force will always push at right angles to the plane of contact between the object and the surface. Both of these features are illustrated in Figure 4.3(a), where a block rests on a flat but sloping surface and is prevented from sliding down the slope by an attached string.

The directions of the corresponding tension $\mathbf{T}$ and normal reaction $\mathbf{N}$ are as shown in Figure 4.3(b).

Although this situation involves three forces, rather than the two in each of the earlier examples, there is still a force balance here, since otherwise the block would start to move.

4.2 Balance of forces

We continue to consider situations in which forces act on an object that does not move. In Subsection 4.1 you saw two such situations where only two forces were involved.

- Where the weight $\mathbf{W}$ of an object is counteracted by the tension $\mathbf{T}$ from a vertical string, the two forces are related by the equation $\mathbf{T} + \mathbf{W} = \mathbf{0}$.
- Where the weight $\mathbf{W}$ of an object is counteracted by the normal reaction $\mathbf{N}$ from a solid horizontal surface, the two forces are related by the equation $\mathbf{N} + \mathbf{W} = \mathbf{0}$.

In each case, the combined effect of the two forces is represented by their vector sum. The fact that this sum is the zero vector indicates that no net force acts, which is a necessary requirement for the object to remain at rest.

These considerations generalise to any other situation in which just two forces act. If an object is acted upon by two forces, $\mathbf{F}$ and $\mathbf{G}$, and remains at rest in the absence of other forces, then

This is sometimes described loosely by saying that $\mathbf{F}$ and $\mathbf{G}$ are 'equal and opposite' forces, meaning that they are equal in magnitude but opposite in direction.

$$\mathbf{F} + \mathbf{G} = \mathbf{0}.$$

This is known as the *Equilibrium Condition* for two forces.

We next seek a corresponding condition where *three* forces are involved, as for example in Figure 4.3. As for the two-force case, we can say the following.

- The combined effect of the forces is given by their vector sum (as stated in Subsection 4.1, this aspect of the vector model of forces is validated well by experiment).
- If the object acted upon by the forces remains at rest, then no net force acts: that is, the combined effect of the forces is the zero vector.

It follows that if an object is acted upon by three forces, $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$, and remains at rest in the absence of other forces, then

$$\mathbf{F} + \mathbf{G} + \mathbf{H} = \mathbf{0}.$$

This is the Equilibrium Condition for three forces. There are various other ways in which it can be expressed. For example, it is equivalent to either of the equations

$$\mathbf{H} = -(\mathbf{F} + \mathbf{G}) \quad \text{or} \quad -\mathbf{H} = \mathbf{F} + \mathbf{G},$$

which say that $\mathbf{H}$ must be 'equal and opposite' to the resultant of $\mathbf{F}$ and $\mathbf{G}$.

By applying the same reasoning, we can now write down a corresponding condition for the general case, in which any number of forces act. This can be expressed concisely using sigma notation for the vector sum.

Sigma notation (for sums of scalars) was introduced in Chapter B1, Subsection 1.2.

Equilibrium Condition for forces

If an object is acted upon by n forces, $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$, and remains at rest in the absence of other forces, then the force vectors satisfy the equation

$$\sum_{i=1}^{n} \mathbf{F}_i = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = \mathbf{0}.$$

In the case of two forces, $\mathbf{F}$ and $\mathbf{G}$, this reduces to $\mathbf{F} + \mathbf{G} = \mathbf{0}$.

In the case of three forces, $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$, it becomes $\mathbf{F} + \mathbf{G} + \mathbf{H} = \mathbf{0}$.

Here we put $\mathbf{F}_1 = \mathbf{F}$, $\mathbf{F}_2 = \mathbf{G}$ and $\mathbf{F}_3 = \mathbf{H}$.

The next example shows how this condition can be applied.

Example 4.1 Finding the third force

An object which remains at rest is acted upon by three forces, $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$, where $\mathbf{F}$ and $\mathbf{G}$ are given in terms of specified Cartesian unit vectors by

$$\mathbf{F} = 7\mathbf{i} + 2\mathbf{j}, \quad \mathbf{G} = -9\mathbf{i} + 3\mathbf{j}.$$

Find the component form of the third force, $\mathbf{H}$.

Solution

Since the object remains at rest, the Equilibrium Condition applies; that is,

$$\mathbf{F} + \mathbf{G} + \mathbf{H} = \mathbf{0}.$$

Hence we find that

$$\begin{aligned} \mathbf{H} &= -\mathbf{F} - \mathbf{G} \\ &= -(7\mathbf{i} + 2\mathbf{j}) - (-9\mathbf{i} + 3\mathbf{j}) \\ &= (-7 + 9)\mathbf{i} + (-2 - 3)\mathbf{j} \\ &= 2\mathbf{i} - 5\mathbf{j}. \end{aligned}$$

Activity 4.1 Finding the third force

An object which remains at rest is acted upon by three forces, $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$, where $\mathbf{G}$ and $\mathbf{H}$ are given in terms of specified Cartesian unit vectors by

$$\mathbf{G} = 2\mathbf{i} + \mathbf{j}, \quad \mathbf{H} = -\mathbf{j}.$$

Find the component form of the force $\mathbf{F}$.

A solution is given on page 63.

What was done in component form in the last example and activity can also be achieved in geometric form if required. Writing the three-force Equilibrium Condition in the form $-\mathbf{H} = \mathbf{F} + \mathbf{G}$, and applying the Triangle Rule to the vector sum $\mathbf{F} + \mathbf{G}$, gives the diagram in Figure 4.4(a) overleaf.

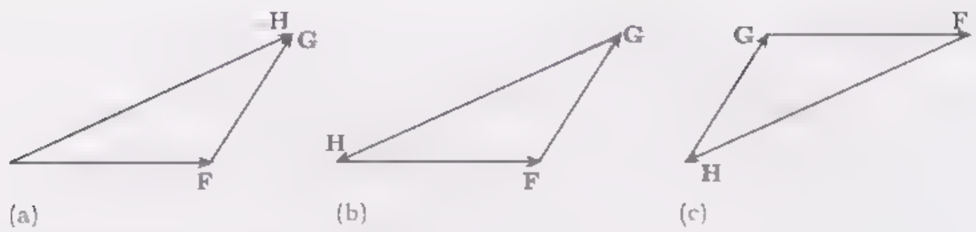


Figure 4.4 (a) Triangle rule for $\mathbf{F} + \mathbf{G}$ (b) Triangle of forces (anticlockwise) (c) Triangle of forces (clockwise)

Figure 4.4(c) shows the equivalent clockwise version of the triangle of forces.

This diagram can be amended by replacing the arrow for the vector $-\mathbf{H}$ with one for $\mathbf{H}$, pointing in the opposite direction, as shown in Figure 4.4(b). This gives a triangle of arrows in which all of the arrows point in the same sense (anticlockwise in this case) around the triangle. This is called the **triangle of forces**. It is the geometric representation of the Equilibrium Condition for three forces.

Given suitable information, such triangles can be solved, in a manner similar to that applied to find the resultants of displacements and velocities in Subsection 3.3. This could require use of the Cosine or Sine Rule but, as in the earlier cases, matters are simplified if the triangle has a right angle.

Example 4.2 Finding unknown magnitudes

An object which remains at rest is acted upon by three forces, $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$. The force $\mathbf{F}$ has magnitude $|\mathbf{F}| = 50\,\text{N}$ and direction $\theta = 30^\circ$ (measured, as usual, anticlockwise from the $\mathbf{i}$-direction). The force $\mathbf{G}$ has direction $\theta = -45^\circ$, and the force $\mathbf{H}$ has direction $\theta = 180^\circ$. By considering the triangle of forces, find correct to one decimal place the magnitudes of the forces $\mathbf{G}$ and $\mathbf{H}$.

Solution

The force diagram and corresponding triangle of forces are shown in Figure 4.5.

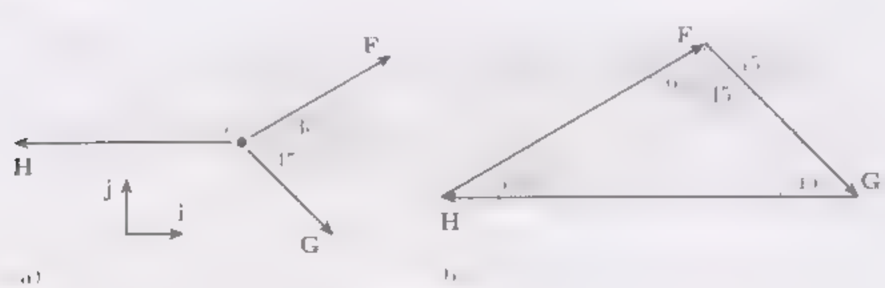


Figure 4.5 (a) Force diagram (b) Triangle of forces (clockwise version)

All of the angles in the triangle of forces can be found from the given force directions, as indicated in Figure 4.5(b), so the Sine Rule can be applied directly, in the form

$$\frac{|\mathbf{F}|}{\sin(45^\circ)} = \frac{|\mathbf{G}|}{\sin(30^\circ)} = \frac{|\mathbf{H}|}{\sin(105^\circ)}.$$

Hence the required force magnitudes are given by

$$|\mathbf{G}| = \frac{|\mathbf{F}|\sin(30^\circ)}{\sin(45^\circ)} = 50 \times \tfrac{1}{2}\sqrt{2} \simeq 35.4\,\mathrm{N},$$

$$|\mathbf{H}| = \frac{|\mathbf{F}|\sin(105^\circ)}{\sin(45^\circ)} \simeq 68.3\,\mathrm{N}.$$

Activity 4.2 Finding unknown magnitudes

An object which remains at rest is acted upon by three forces, **F**, **G** and **H**. The force **G** has magnitude $|\mathbf{G}| = 70\,\mathrm{N}$ and direction $\theta = 0$. The force **F** has direction $\theta = 90^\circ$, and the force **H** has direction $\theta = -120^\circ$. By considering the triangle of forces, find the magnitudes of the forces **F** and **H**, giving your answers as exact values.

Solutions are given on page 64.

In the next section, the Equilibrium Condition will be applied to further problems involving three or four forces. For three forces, either the component approach or the triangle of forces may reasonably be applied, but where there are more than three forces, the use of components is usually preferable.

Summary of Section 4

This section has introduced:

- the mass of an object;
- the particle model for objects;
- the modelling of forces by vectors;
- three specific types of force, namely, weight, tension and normal reaction;
- force diagrams;
- the Equilibrium Condition for the forces acting on an object that remains at rest;
- the triangle of forces.

Exercises for Section 4

Exercise 4.1

An object which remains at rest is acted upon by three forces, **P**, **Q** and **R**, where **P** and **Q** are given in terms of specified Cartesian unit vectors by

$$\mathbf{P} = 3\mathbf{i} - 6\mathbf{j}, \quad \mathbf{Q} = 5\mathbf{i} + 2\mathbf{j}.$$

Find the component form of the force **R**.

Exercise 4.2

An object which remains at rest is acted upon by three forces, **P**, **Q** and **R**. The force **P** has magnitude $|\mathbf{P}| = 120\,\mathrm{N}$ and direction $\theta = -50^\circ$. The force **Q** has direction $\theta = 20^\circ$, and the force **R** has direction $\theta = 150^\circ$. By considering the triangle of forces, find the magnitudes of the forces **Q** and **R**, giving your answers correct to one decimal place

5 Analysing force problems

To study Subsection 5.1 you will need a cassette player and Audio Tape 2.

This section looks largely at the application of previously developed techniques to statics problems. Subsection 5.1 analyses a couple of specific examples in detail. Subsection 5.2 adds to the types of force met so far by introducing a simple model for *friction*.

5.1 Three-force problems

In the audio tape which follows, you will see the methods of Subsection 4.2 brought to bear on two problems of a more 'real-world' nature than those considered so far. These methods are based on applying the Equilibrium Condition for the forces that act on an object at rest, and involve use of either the components of forces or the triangle of forces.

Now listen to Audio Tape 2, Band 2, 'Applying the vector model of forces'.

Frame 1

The Great Pyramid of Cheops (Khufu) at Giza

Cheops comes from the Greek form of Pharaoh Khufu's name.

2.5 million blocks of limestone, 2.6 tonnes each.

How were these blocks raised?

Frame 2

A specific problem

A block of stone, of mass 2600 kg, is held at rest on a frictionless ramp by workers pulling on a rope attached to the block. The ramp makes an angle of 10° with the horizontal. Each worker can pull with a force of magnitude at most 350 N. What is the minimum number of workers required to achieve this?

Take $g = 10\,\text{m s}^{-2}$.

'frictionless' = 'smooth' = no resistance to motion via surface contact

Modelling assumptions?

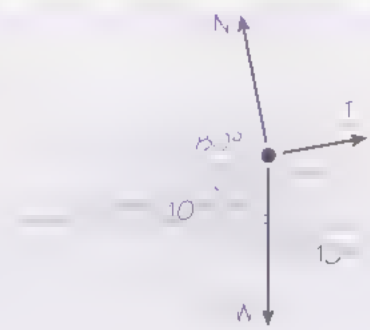
Model as a particle.

*/ for 3, 4, 7 see Step press!!

What forces act?

The forces acting on the particle are:

- $\underline{W}$ the weight of the block;
- $\underline{N}$ the normal reaction from the ramp;
- $\underline{T}$ the tension in the rope.

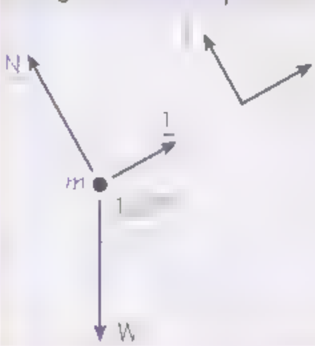


force diagram

No motion ⟶ Equilibrium Condition applies: net force acting is zero, i.e.

$$\underline{W} + \underline{N} + \underline{T} = \underline{0}$$

Using vector components



In newtons,

$|\underline{W}| = mg$
$= 2600 \times 10.$

$$\underline{W} = |\underline{W}|\cos(-100^\circ)\,\underline{i} + |\underline{W}|\sin(-100^\circ)\,\underline{j}$$
$$= 26000\cos(-100^\circ)\,\underline{i} + 26000\sin(-100^\circ)\,\underline{j}$$

Equilibrium Condition in components

$\underline{W} + \underline{N} + \underline{T} = \underline{0} \longrightarrow 26000\cos(-100^\circ)\,\underline{i} + 26000\sin(-100^\circ)\,\underline{j} + |\underline{N}|\,\underline{j} + |\underline{T}|\,\underline{i} = \underline{0}.$

Equate $\underline{i}$- and $\underline{j}$-components separately to zero.

$\underline{i}$: $26000\cos(-100^\circ) + |\underline{T}| = 0$; $|\underline{T}| = 4514.85$

$\underline{j}$: $26000\sin(-100^\circ) + |\underline{N}| = 0$; $|\underline{N}| = 25605.01$

So how many workers does it take?

How many workers?

Magnitude of normal reaction is

$$|\underline{N}| = -26000 \sin(-100^\circ) \simeq 25605 \text{ newtons.}$$

Magnitude of tension in rope is

$$|\underline{T}| = -26000 \cos(-100^\circ) \simeq 4515 \text{ newtons.}$$

Each worker pulls with force magnitude of 350N at most.

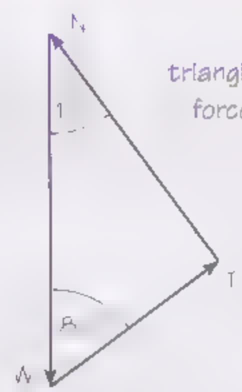
$$\frac{4515}{350} \simeq 12.9.$$

But this ignores presence of friction . . .

Hence $\boxed{13}$ workers are required to keep block at rest.

Frame 7

Using the triangle of forces



Now use trigonometry.

$$
\begin{aligned}
|\underline{T}| &= |\underline{W}| \sin(10^\circ) \\
&= mg \sin(10^\circ) \\
&= 26000 \sin(10^\circ) \\
&\simeq 4515 \text{ newtons.}
\end{aligned}
$$

Easy! So why use components?

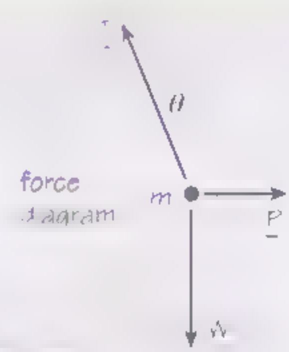
Another context – demolition

A wrecking ball, of mass 2000 kg, is suspended from a chain of length 6 metres whose other end is held fixed. If the ball is subjected also to a horizontal force of magnitude 350 N, and then stays at rest, what is the horizontal distance of the ball from its position with the chain vertical?

Forces acting

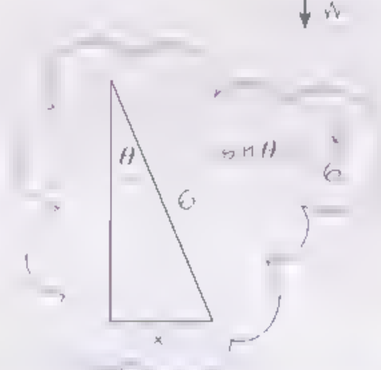
The forces acting on the particle are:

- $\underline{W}$ the weight of the ball;
- $\underline{T}$ the tension in the chain;
- $\underline{P}$ the horizontal pushing force.



No motion ⟶ Equilibrium Condition applies:

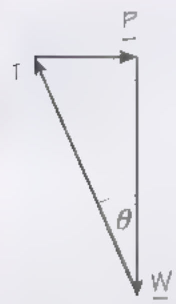
$$\boxed{\underline{W} + \underline{T} + \underline{P} = \underline{0}\,.}$$



Use triangle of forces to find θ ; then find x .

Take $g = 10\,\text{m s}^{-2}$.

Applying the triangle of forces



triangle
of forces

$$\tan\theta = \frac{|\underline{P}|}{|\underline{W}|} = \frac{P}{mg} = \frac{350}{20000} = 0.0175$$

so $\theta \simeq$ [illegible] and $\sin\theta \simeq 0.0175$

For small θ, the values of $\sin\theta$ and $\tan\theta$ are very close.

Hence

$$x \simeq 6 \times 0.0175$$
$$\simeq 0.10 \text{ metres.}$$

So you can push the ball only about 10 cm from its original equilibrium position!

Summary

- Object is modelled by a particle (two examples).
- Particle is acted upon by forces, represented by vectors.
- Particle at rest ⟶ Equilibrium Condition applies: vector sum of forces is zero.
- Then two possible approaches.

  (a) Express forces and Equilibrium Condition in terms of components; turns vector equation into two scalar equations.

  (b) Draw triangle of forces and then use trigonometry.

need to choose convenient unit vectors $\underline{i}$, $\underline{j}$

applies however many forces act

works only when there are 3 forces

The next pair of activities refer to the two situations which were analysed on the audio tape.

Activity 5.1 The pyramid ramp problem revisited

In the course of solving the problem stated in Frame 2, the Cartesian unit vectors were chosen (in Frame 4) with $\mathbf{i}$ parallel to the slope of the ramp and $\mathbf{j}$ at right angles to it. Solve the problem anew, using components referred to Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$ which are respectively horizontal and vertical.

A solution is given on page 64.

Comment

This demonstrates that the particular choice made for the Cartesian unit vectors does not affect the outcome of the analysis. However, it also shows that the choice of $\mathbf{i}$ and $\mathbf{j}$ made in Frame 4 leads to an easier calculation.

Activity 5.2 The wrecking ball problem revisited

The problem stated in Frame 8 was solved using the triangle of forces. Solve the problem anew, using components. (Choose the Cartesian unit vectors so that two of the three forces in the problem will have only one non-zero component each.)

A solution is given on page 64.

The remaining two activities ask you to analyse situations not considered previously.

Activity 5.3 Forces on a flower basket

A hanging flower basket of mass 4 kg is attached to two cords (see Figure 5.1). One cord makes an angle of 60° with the horizontal and is also attached to a fixed point on the ceiling. The other cord is horizontal and is also attached to a fixed point on a wall. The flower basket remains at rest. Find the magnitudes of the tensions from the two cords, by

(a) using components. (b) using the triangle of forces.

Take $g = 10\,\mathrm{m\,s^{-2}}$, and give your answers correct to one decimal place.

Solutions are given on page 65.

Figure 5.1 A hanging basket

Activity 5.4 Forces on a supermarket trolley

A full supermarket trolley of mass 60 kg is on a ramp at 15° to the horizontal. The trolley is held at rest by a pulling force $\mathbf{P}$, whose direction makes an angle of 40° with the horizontal. Assuming that the only forces to act on the trolley are $\mathbf{P}$, the weight of the trolley and the normal reaction from the ramp, use the triangle of forces to find the magnitude of the pulling force, to the nearest newton. Take $g = 10\,\mathrm{m\,s^{-2}}$.

A solution is given on page 65.

5.2 Modelling friction

If an object is pushed or pulled sideways across a flat surface, then some resistance to motion is encountered. If the push or pull is not strong enough, then the object will not start to move at all. This resistance is due to static *friction* between the object and the surface on which it rests.

This subsection will not be assessed.

Experimental evidence suggests that this force may be modelled as a vector $\mathbf{F}$, such that:

- the direction of $\mathbf{F}$ is parallel to the surface on which the object lies and opposite to the direction of motion which would occur in the absence of $\mathbf{F}$;
- the magnitude $|\mathbf{F}|$ is just sufficient to prevent motion, up to a maximum value of $\mu|\mathbf{N}|$, where $\mathbf{N}$ is the normal reaction from the surface on the object and μ is a constant called the *coefficient of static friction*.

If the value of $|\mathbf{F}|$ required to prevent motion is greater than $\mu|\mathbf{N}|$, then the object moves.

The coefficient μ is a property of the materials which form the object and flat surface, with a higher value of μ corresponding to a 'rougher' surface. For example, the coefficient of static friction for oil-lubricated contact between steel and steel is $\mu \simeq 0.1$, whereas for rubber on tarmac the coefficient is $\mu \simeq 1.3$.

The symbol μ is the Greek letter 'mu'.

For a hypothetical 'smooth' (frictionless) surface, we have $\mu = 0$.

The pyramid ramp problem once more

As an application of this model of friction, we shall show how the conclusion of Frame 6 in Subsection 5.1 may be revised to take account of friction.

First, we need an estimate for the coefficient of static friction, μ, which applies for contact between the sledge containing a stone block and the lubricated ramp. For this purpose, we refer to an Egyptian relief picture which shows in detail the transport of a huge alabaster statue on a sledge across horizontal ground. Liquid is being poured from pots in front of the statue, in order to reduce friction, and precisely 172 men are depicted on the ropes with which the statue is being pulled along. The mass of the statue has been estimated as 60 tonnes, that is, 60 000 kg.

This picture is at the tomb of the XIIth dynasty noble Dhutihetop.

The force diagram for this situation is shown in Figure 5.2, where $\mathbf{W}$ is the weight of the statue, $\mathbf{N}$ is the normal reaction, $\mathbf{T}$ is the tension force caused by the pulling of 172 men, and $\mathbf{F}$ is the opposing force of friction.

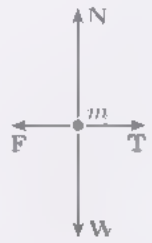


Figure 5.2 Force diagram for transport of huge statue

Assume that each man can pull with a force magnitude 350 N (as in Subsection 5.1), and that their combined maximum effort is *just* sufficient to overcome static friction and move the statue. On writing down the (approximate) Equilibrium Condition and then taking components, we obtain

$$|\mathbf{N}| = |\mathbf{W}| = mg \simeq 600\,000\,\mathrm{N},$$
$$|\mathbf{F}| \simeq |\mathbf{T}| = 172 \times 350 = 60\,200\,\mathrm{N}.$$

Also, the model of friction gives $|\mathbf{F}| = \mu|\mathbf{N}|$. It follows that

$$60\,200 \simeq 600\,000\mu; \quad \text{that is,} \quad \mu \sim 0.1.$$

Having obtained a value for μ, consider once more the 2600 kg stone block being held on a 1-in-6 ramp, from Frames 2–6 of Subsection 5.1. We found (in Frame 6) that $|\mathbf{N}| \simeq 25\,605$ N for this situation. This corresponds to a friction force of maximum magnitude

$$|\mathbf{F}| = \mu|\mathbf{N}| \simeq 0.1 \times 25\,605 \simeq 2560\,\mathrm{N}.$$

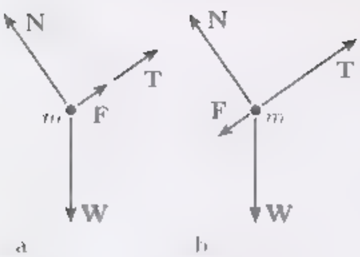


Figure 5.3 Force diagram for stone block and sledge, with friction acting: (a) up the slope; (b) down the slope

This is the pulling force provided by $2560/350 \simeq 7.3$ men. Since we calculated previously that 12.9 men were needed to keep the block at rest on a *frictionless* ramp, it follows that $12.9 - 7.3 = 5.6$ men (or 6, for a whole number of men) would suffice to stop the block sliding down the ramp. In this case, friction acts *up* the slope, as shown in Figure 5.3(a), to oppose sliding downwards.

However, if the aim is to drag the block up the slope, then the pulling force must overcome the effect of friction acting *down* the slope, as shown in Figure 5.3(b). This will take the efforts of at least $12.9 + 7.3 = 20.2$ men, that is, 21 men.

This is the conclusion of our revised model for the problem of Frame 2, where now friction has been taken into account.

Summary of Section 5

This section has demonstrated in several cases how to apply the Equilibrium Condition for forces acting on an object at rest, using either components or the triangle of forces (see Frame 11).

Exercise for Section 5

Exercise 5.1

A block of wood of mass 0.6 kg is at rest on a smooth flat surface at an angle of 45° to the horizontal.

(a) The block is pushed horizontally, to prevent it from sliding down the slope. Using components, and taking $g = 10\,\mathrm{m\,s^{-2}}$, find the magnitude of this pushing force.

(b) The pushing force is now changed, so that it is directed upwards at 30° to the horizontal, and its magnitude is adjusted so that the block remains at rest. Use the triangle of forces to find the magnitude of the pushing force in this case, giving your answer correct to one decimal place.

Summary of Chapter B3

In this chapter you saw how vectors, which were introduced as columns of numbers in Chapter B2, can be represented also in geometric and component form. This greatly widens their applicability as models for physical phenomena, and they have been applied here to situations which involve displacement, velocity and force.

The geometric form lends itself better to visualisation and direct interpretation in these applications, while component form provides an easier way of combining vectors. Hence it is important to be able to convert between these forms of a vector, though vectors in geometric form can also be added by applying the Sine Rule or Cosine Rule.

Learning outcomes

You have been working towards the following learning outcomes.

Terms to know and use

Scalar, vector (in column, geometric and component form), components of a vector, magnitude and direction, Cartesian unit vectors, addition, scalar multiplication and subtraction of vectors, Triangle Rule, Parallelogram Rule, resultant, zero vector, displacement (vector), position vector, bearing, velocity (vector), speed, Sine Rule, Cosine Rule, force, mass, particle, weight, tension, normal reaction, force diagram, Equilibrium Condition, triangle of forces, friction.

Symbols and notation to know and use

$\mathbf{a}$ (in print, represented in handwriting by $\underset{\sim}{a}$ or $\underline{a}$), $\mathbf{i}$, $\mathbf{j}$, $\mathbf{0}$;

$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1\mathbf{i} + a_2\mathbf{j}$, $|\mathbf{a}|$ for magnitude of $\mathbf{a}$, θ, $\overrightarrow{PQ}$;

$\mathbf{F}$, $\mathbf{W}$, $\mathbf{T}$, $\mathbf{N}$, m, g.

Mathematical skills

- Add, subtract and scalar multiply vectors in column or component form.
- Sketch the outcomes of addition, subtraction and scalar multiplication for vectors in geometric form, including use of the Triangle Rule or Parallelogram Rule to find the resultant of two vectors.
- Given the coordinates of points P and Q, write down the displacement vector $\overrightarrow{PQ}$ in either column or component form.
- Given a vector in component (or column) form, find its geometric form (in terms of magnitude and direction).
- Given a vector in geometric form, find its components.
- Use bearings to indicate directions.
- Use the Sine Rule and Cosine Rule to solve triangles.
- Apply the Equlibrium Condition to find information about force vectors.

Modelling skills

- Use vectors to model displacements, velocities or forces, and interpret vector results in terms of the original situation.
- Use the particle model of objects where appropriate, so that all forces acting on an object can be considered to act at the same point.
- Choose Cartesian unit vectors so as to simplify the component forms of force vectors as far as possible.
- Invoke the Equilibrium Condition as a mathematical statement that the forces acting on an object which remains at rest are in balance.

Ideas to be aware of

- Vectors can be represented in different forms, and which of the forms is the most appropriate to use depends on the application.
- Informally, vectors are quantities which have magnitude and direction. They add geometrically according to the Triangle Rule, which is equivalent to the addition of corresponding components.

Summary of Block B

This block has developed the theme of mathematical modelling, while showing further applications of recurrence sequences. The corresponding models are called discrete, indicating that the independent variable takes isolated rather than continuously varying values. In Block C, by contrast, the focus will be entirely on continuous models.

Chapter B1 looked at models for populations. The recurrence relation for the first of these (the exponential model) was linear, and hence it was possible to find a closed-form solution, whereas this could not be done for the second (logistic) model. Here we resorted to reasoning about the form of the recurrence relation and to numerical and graphical investigation in order to see what behaviour the model predicted. This led on to wider consideration of the long-term behaviour of sequences, based on the ideas of convergence and limits. These latter ideas also have central importance in the calculus topics of Block C.

The terms of the sequences considered in Chapter B2 were vectors (columns of numbers), which permitted the construction of a simple model to reflect the age structure of a population. The coefficients in the corresponding recurrence relations were matrices, for which there is an operation of multiplication that proves extremely useful as part of this population model and many other models.

Chapter B3 concentrated again on vectors, showing that they can also be described in geometric form and have broader modelling application than the column form would suggest.

This is an instructive development from a more general point of view. It demonstrates that a mathematical concept, such as 'vector', can sometimes be translated from the context in which it arose to a superficially different but mathematically equivalent form. In the new form, alternative modelling possibilities arise, while some of the features of the original form can profitably be applied to the new modelling contexts.

Solutions to Activities

Solution 1.1

The vector sums are as follows.

(a) $\begin{pmatrix}4\\-2\end{pmatrix}+\begin{pmatrix}-3\\1\end{pmatrix}=\begin{pmatrix}4-3\\-2+1\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}$

[illegible] $\begin{pmatrix}5\\3\end{pmatrix}+\begin{pmatrix}[illegible]\\-3\end{pmatrix}=\begin{pmatrix}5-4\\3-3\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$

[illegible] $\begin{pmatrix}-7\\[illegible]\end{pmatrix}+\begin{pmatrix}2\\[illegible]\end{pmatrix}+\begin{pmatrix}5\\[illegible]\end{pmatrix}=\begin{pmatrix}-7+2+5\\[illegible]\end{pmatrix}=\begin{pmatrix}0\\[illegible]\end{pmatrix}$

Solution 1.2

The scalar multiples are as follows.

(a) $4\mathbf{a}=4\begin{pmatrix}2\\-1\end{pmatrix}=\begin{pmatrix}8\\-4\end{pmatrix}$

(b) $-\mathbf{a}=(-1)\begin{pmatrix}2\\-1\end{pmatrix}=\begin{pmatrix}-2\\1\end{pmatrix}$

[illegible] $\frac{1}{2}\mathbf{a}=\frac{1}{2}\begin{pmatrix}2\\-1\end{pmatrix}=\begin{pmatrix}1\\-\frac{1}{2}\end{pmatrix}$

Solution 1.3

The combined vector operations are as follows.

(a) $2\begin{pmatrix}6\\-3\end{pmatrix}-7\begin{pmatrix}1\\2\end{pmatrix}+5\begin{pmatrix}[illegible]\\4\end{pmatrix}$

$=\begin{pmatrix}2\times 6-7\times 1+5\times(-1)\\2\times(-3)-7\times 2+5\times 4\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

[illegible] $a_1\begin{pmatrix}1\\0\end{pmatrix}+a_2\begin{pmatrix}0\\1\end{pmatrix}$

$=\begin{pmatrix}a_1\times 1+a_2\times 0\\a_1\times 0+a_2\times 1\end{pmatrix}=\begin{pmatrix}a_1\\a_2\end{pmatrix}$

Solution 1.4

The arrows are shown in Figure S.1. Note that it does not matter where in the plane they are placed.

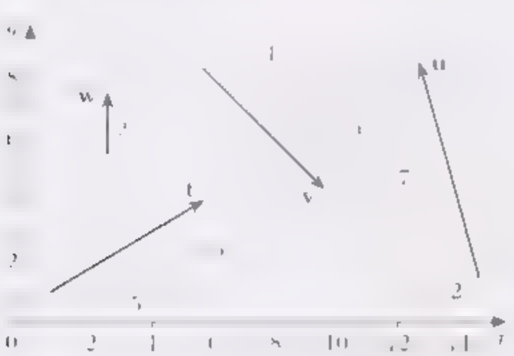


Figure S.1

For the vector $\mathbf{t}$, we have $t_1 = 5 > 0$ and $t_2 = 3 > 0$, so the arrow must point in a direction for which both x and y increase. With the standard configuration for the x- and y-axes, this direction for $\mathbf{t}$ is 'to the right and up'.

Similarly, for the vector $\mathbf{u}$, we require an arrow that points in a direction for which x decreases ($u_1 = -2 < 0$) and y increases ($u_2 = 7 > 0$); that is, $\mathbf{u}$ points 'to the left and up'

The vector $\mathbf{v}$ has components $v_1 = 4 > 0$ and [illegible] < 0 so points 'to the right and down'

The vector $\mathbf{w}$ has first component $w_1 = 0$, so points vertically. Since $w_2 > 0$, $\mathbf{w}$ points upwards

Solution 1.5

The required vector magnitudes are:

$$|\mathbf{p}| = \sqrt{2^2+2^2}=\sqrt{8}=2\sqrt{2};$$

$$|\mathbf{r}| = \sqrt{(-1)^2+(-1)^2}=\sqrt{2};$$

$$|\mathbf{s}| = \sqrt{(-2)^2+(2\sqrt{3})^2}=\sqrt{16}=4.$$

Solution 1.6

The solution is illustrated by Figure S.2. In each case, the arrow lies along the hypotenuse of a right-angled triangle, and the size of an angle that the arrow makes with [illegible] direction can be deduced from the lengths of the other two sides of the triangle.

For the vector $\mathbf{p}$, the angle that the arrow makes with the positive x-direction has size 45°, because [illegible] $= 1$. Since the arrow points 'to the right and up', the direction of $\mathbf{p}$ is given by $\theta = 45°$.

For the vector $\mathbf{r}$, the angle that the arrow makes with the negative x-direction has size 45°, because $\tan(45°) = 1$. Since the arrow points 'to the left and down', the direction of $\mathbf{r}$ is given by $\theta = -180° + 45° = -135°$.

For the vector $\mathbf{s}$, the angle that the arrow makes with the negative x-direction has size 60°, because $\tan(60°) = \sqrt{3}$. Since the arrow points 'to the left and up', the direction of $\mathbf{s}$ is given by $\theta = 180° - 60° = 120°$

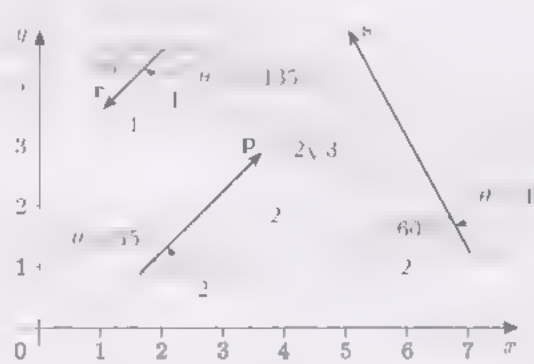


Figure S.2

Solution 1.7

The vector sums are represented by the arrows labelled $\mathbf{p}+\mathbf{q}$ and $\mathbf{r}+\mathbf{s}$ in Figure S.3.

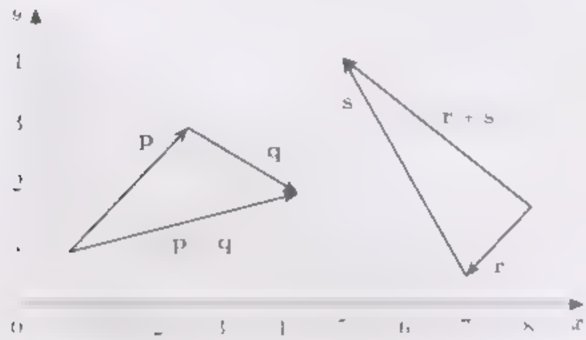


Figure S.3

Solution 1.8

Arrows to represent the scalar multiples are sketched in Figure S.4.

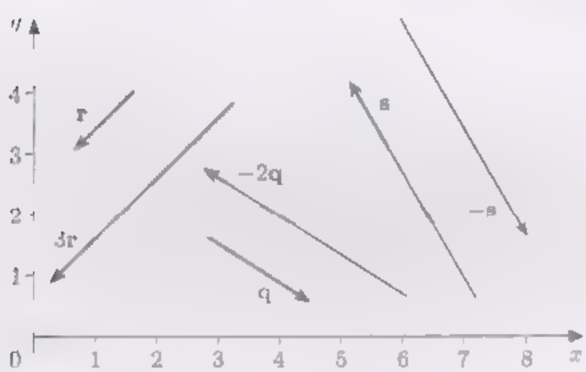


Figure S.4

Solution 1.9

The construction of arrows to represent the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ is shown in Figure S.5.

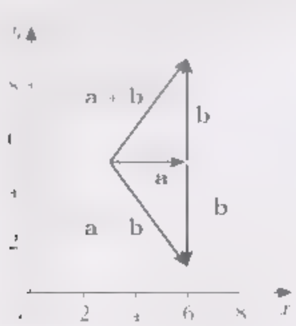


Figure S.5

Solution 1.10

The vector $-2\mathbf{t}+3\mathbf{u}+4\mathbf{v}$ is equal to

$$\begin{aligned}
&-2(5\mathbf{i}+3\mathbf{j})+3(-2\mathbf{i}+7\mathbf{j})+4(4\mathbf{i}-4\mathbf{j})\\
&=(-2\times 5+3\times(-2)+4\times 4)\mathbf{i}\\
&\quad+(-2\times 3+3\times 7+4\times(-4))\mathbf{j}\\
&=0\mathbf{i}+(-1)\mathbf{j}\\
&=-\mathbf{j}.
\end{aligned}$$

Hence the vector $-2\mathbf{t}+3\mathbf{u}+4\mathbf{v}$ has $\mathbf{i}$-component 0 and $\mathbf{j}$-component -1.

Solution 2.1

In each case, we apply the formula

$$\mathbf{a}=|\mathbf{a}|\cos\theta\,\mathbf{i}+|\mathbf{a}|\sin\theta\,\mathbf{j}.$$

(a) (i) Here $|\mathbf{a}|=3$ and $\theta=110^\circ$, so the component form is

$$\begin{aligned}
\mathbf{a}&=3\cos(110^\circ)\mathbf{i}+3\sin(110^\circ)\mathbf{j}\\
&=-1.0261\mathbf{i}+2.8191\mathbf{j}\quad\text{(to 4 d.p.)}.
\end{aligned}$$

(ii) Here $|\mathbf{a}|=2.5$ and $\theta=-20^\circ$, so the component form is

$$\begin{aligned}
\mathbf{a}&=2.5\cos(-20^\circ)\mathbf{i}+2.5\sin(-20^\circ)\mathbf{j}\\
&=2.3492\mathbf{i}-0.8551\mathbf{j}\quad\text{(to 4 d.p.)}.
\end{aligned}$$

(b) The results for part (b) are illustrated in Figure S.6.

(i) (ii)

Figure S.6

(i) Here $|\mathbf{a}|=1$ and $\theta=45^\circ$, so the component form is

$$\mathbf{a}=1\cos(45^\circ)\mathbf{i}+1\sin(45^\circ)\mathbf{j}=\tfrac{1}{2}\sqrt{2}\mathbf{i}+\tfrac{1}{2}\sqrt{2}\mathbf{j}.$$

(ii) Here $|\mathbf{a}|=5$ and $\theta=-150^\circ$. Using in turn the trigonometric identities (from Chapter A2)

$$\begin{aligned}
&\cos(-\theta)=\cos\theta,\\
&\sin(-\theta)=-\sin\theta,\\
&\cos(180^\circ-\theta)=-\cos\theta,\\
&\sin(180^\circ-\theta)=\sin\theta,
\end{aligned}$$

we obtain the component form

$$\begin{aligned}
\mathbf{a}&=5\cos(-150^\circ)\mathbf{i}+5\sin(-150^\circ)\mathbf{j}\\
&=5\cos(150^\circ)\mathbf{i}-5\sin(150^\circ)\mathbf{j}\\
&=-5\cos(30^\circ)\mathbf{i}-5\sin(30^\circ)\mathbf{j}\\
&=-5\times\tfrac{1}{2}\sqrt{3}\mathbf{i}-5\times\tfrac{1}{2}\mathbf{j}\\
&=-\tfrac{5}{2}\sqrt{3}\mathbf{i}-\tfrac{5}{2}\mathbf{j}
\end{aligned}$$

Solution 2.2

Arrows to represent the four vectors are shown in Figure S.7.

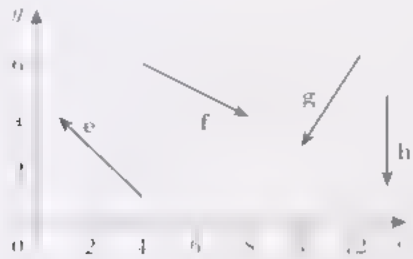


Figure S.7

(a) The magnitude of the vector $\mathbf{e} = -3\mathbf{i} + 3\mathbf{j}$ is

$$|\mathbf{e}| = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}.$$

Since the components of $\mathbf{e}$ are $e_1 = -3$, $e_2 = 3$, we have

$$\phi = \arctan|3/(-3)| = \arctan 1 = 45^\circ.$$

Also, $(-3, 3)$ lies in the second quadrant, so the direction of $\mathbf{e}$ is $\theta = 180^\circ - \phi = 135^\circ$.

(b) The magnitude of the vector $\mathbf{f} = 4\mathbf{i} - 2\mathbf{j}$ is

$$|\mathbf{f}| = \sqrt{4^2 + (-2)^2} = \sqrt{20} = 2\sqrt{5}.$$

Since the components of $\mathbf{f}$ are $f_1 = 4$, $f_2 = -2$, we have

$$\phi = \arctan|-2/4| = \arctan \tfrac{1}{2} \simeq 26.6^\circ.$$

Also, $(4, -2)$ lies in the fourth quadrant, so the direction of $\mathbf{f}$ is $\theta = -\phi \simeq -26.6^\circ$.

(c) The magnitude of the vector $\mathbf{g} = -2\mathbf{i} - 3\mathbf{j}$ is

$$|\mathbf{g}| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}.$$

Since the components of $\mathbf{g}$ are $g_1 = -2$, $g_2 = -3$, we have

$$\phi = \arctan|-3/(-2)| = \arctan \tfrac{3}{2} \simeq 56.3^\circ.$$

Also, $(-2, -3)$ lies in the third quadrant, so the direction of $\mathbf{g}$ is $\theta = -(180^\circ - \phi) \simeq -123.7^\circ$.

(d) The magnitude of the vector $\mathbf{h} = -3.5\mathbf{j}$ is

$$|\mathbf{h}| = \sqrt{(-3.5)^2} = 3.5.$$

This vector has the form $\mathbf{h} = h_2\mathbf{j}$, so its direction can be found directly from Figure 2.4. Since $h_2 = -3.5 < 0$, the direction is $\theta = -90^\circ$.

Solution 2.3

Adopting the labels $\mathbf{p}$ and $\mathbf{q}$ for the two given vectors, you found in Activity 2.1(a) that the component forms were (to 4 d.p.)

$$\mathbf{p} = -1.0261\mathbf{i} + 2.8191\mathbf{j},$$
$$\mathbf{q} = 2.3492\mathbf{i} - 0.8551\mathbf{j}.$$

Their sum is

$$\mathbf{p} + \mathbf{q} = (-1.0261 + 2.3492)\mathbf{i} + (2.8191 - 0.8551)\mathbf{j}$$
$$= 1.3231\mathbf{i} + 1.9640\mathbf{j}.$$

The magnitude of the vector $\mathbf{r} = \mathbf{p} + \mathbf{q}$ is

$$|\mathbf{r}| = \sqrt{(1.3231)^2 + (1.9640)^2} \simeq 2.37.$$

Since the components of $\mathbf{r}$ are $r_1 = 1.3231$, $r_2 = 1.9640$, we have

$$\phi = \arctan(1.9640/1.3231) \simeq 56.03^\circ.$$

Also, $(1.3231, 1.9640)$ lies in the first quadrant, so the direction of $\mathbf{r}$ is $\theta = \phi \simeq 56.03^\circ$.

Solution 2.4

(a) The directions are shown in Figure S.8.

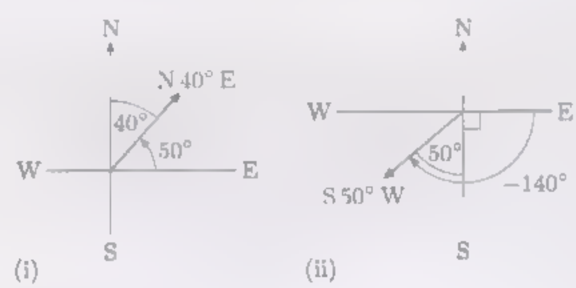


Figure S.8

(i) The direction is $\theta = 90^\circ - 40^\circ = 50^\circ$.

(ii) The direction is $\theta = -(90^\circ + 50^\circ) = -140^\circ$.

(b) The directions are shown in Figure S.9.

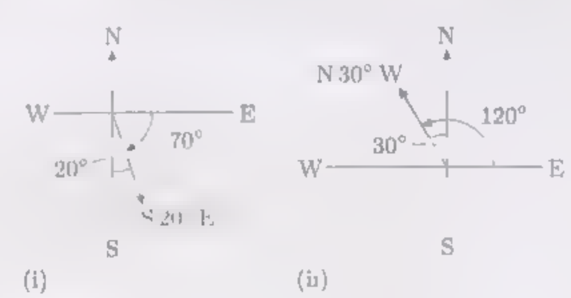


Figure S.9

(i) The bearing is S 20°E.

(ii) The bearing is N 30°W.

Solution 2.5

(a) The strategy on page 22 may be applied. The magnitude of the vector **d** = 77**i** + 286**j** is

$$|\mathbf{d}| = \sqrt{77^2 + 286^2} \simeq 296.$$

Since the components are $d_1 = 77$, $d_2 = 286$, we have

$$\phi = \arctan(286/77) \simeq 75^\circ.$$

Also, (77, 286) lies in the first quadrant, so the direction of **d** is $\theta = \phi \simeq 75^\circ$. This direction corresponds to the bearing N 15°E. Thus **d** describes the displacement 296 km at N 15°E, as claimed.

(b) If **e** is the displacement vector from Exeter to Belfast, then $|\mathbf{e}| = 465$. The direction of **e** is $\theta = 90^\circ + 21^\circ = 111^\circ$. Hence the component form is

$$\begin{aligned} \mathbf{e} &= |\mathbf{e}|\cos\theta\,\mathbf{i} + |\mathbf{e}|\sin\theta\,\mathbf{j} \\ &= 465\cos(111^\circ)\mathbf{i} + 465\sin(111^\circ)\mathbf{j} \\ &\simeq -167\mathbf{i} + 434\mathbf{j} \end{aligned}$$

(Note that we have extended use of the $\simeq$ symbol to an approximate vector equation.)

(i) The **i**-component, −167, gives in kilometres the distance by which Belfast is East of Exeter. However, the minus sign indicates that Belfast is 167 km *West* of Exeter.

(ii) The **j**-component shows that Belfast is 434 km North of Exeter.

(c) The displacement from Belfast to Exeter is the opposite of that from Exeter to Belfast. Hence it is described by the vector −**e**, where **e** was given in part (b).

(i) In component form, the displacement vector from Belfast to Exeter is

$$-\mathbf{e} \simeq 167\mathbf{i} - 434\mathbf{j}$$

(ii) The bearing opposite to N 21°W is S 21°E. Hence, in terms of direct distance and a bearing, the displacement from Belfast to Exeter is 465 km at S 21°E.

Solution 2.6

Let **i** be 1 km East and let **j** be 1 km North. Denote the displacement from Exeter to Belfast by the vector **a**, and the displacement from Belfast to Edinburgh by the vector **b**. These vectors are sketched in Figure S.10.

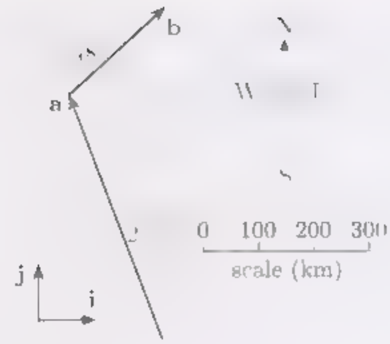


Figure S.10

You showed in Activity 2.5(b) that

$$\mathbf{a} \simeq -167\mathbf{i} + 434\mathbf{j}$$

Similarly, the vector **b** has magnitude $|\mathbf{b}| = 230$ and direction $90^\circ - 48^\circ = 42^\circ$, so its component form is

$$\begin{aligned} \mathbf{b} &= 230\cos(42^\circ)\mathbf{i} + 230\sin(42^\circ)\mathbf{j} \\ &\simeq 171\mathbf{i} + 154\mathbf{j}. \end{aligned}$$

The resultant is

$$\mathbf{c} = \mathbf{a} + \mathbf{b} \simeq (171 - 167)\mathbf{i} + (434 + 154)\mathbf{j} = 4\mathbf{i} + 588\mathbf{j}$$

The strategy on page 22 may now be applied. The vector **c** has magnitude

$$|\mathbf{c}| \simeq \sqrt{4^2 + 588^2} \simeq 588.$$

Since the components are $c_1 \simeq 4$, $c_2 \simeq 588$, we have

$$\phi \simeq \arctan(588/4) \simeq 89.6^\circ.$$

Also, (4, 588) lies in the first quadrant, so the direction of **c** is $\theta = \phi \simeq 89.6^\circ$, which is 90° to the nearest degree. This corresponds to the bearing due North.

Hence the displacement from Exeter to Edinburgh is 588 km due North.

Solution 2.7

The wind blows *from* the South-West, and hence moves *towards* the North-East. Hence the wind velocity vector **v** has direction $\theta = 45^\circ$ (measured anticlockwise from the **i**-direction). Its component form is

$$\begin{aligned} \mathbf{v} &= 2\cos(45^\circ)\mathbf{i} + 2\sin(45^\circ)\mathbf{j} \\ &= 2 \times \tfrac{1}{2}\sqrt{2}\mathbf{i} + 2 \times \tfrac{1}{2}\sqrt{2}\mathbf{j} \\ &= \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} \end{aligned}$$

Solution 2.8

Let $\mathbf{i}$ be $1\,\mathrm{m\,s^{-1}}$ East and let $\mathbf{j}$ be $1\,\mathrm{m\,s^{-1}}$ North. Suppose that $\mathbf{v}_s$ is the velocity of the ship in still water and $\mathbf{v}_c$ is the velocity of the current. These vectors are shown in Figure S.11.

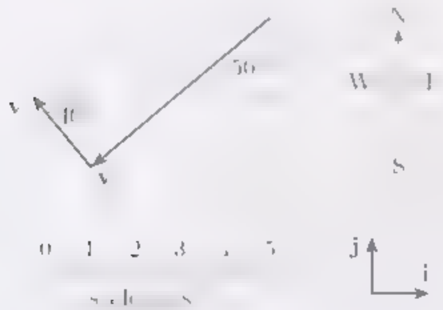


Figure S.11

From the information given, $\mathbf{v}_s$ has magnitude $|\mathbf{v}_s| = 5$ and direction $\theta_s = -(90^\circ + 50^\circ) = -140^\circ$, while $\mathbf{v}_c$ has magnitude $|\mathbf{v}_c| = 2$ and direction $\theta_c = 90^\circ + 40^\circ = 130^\circ$.

The component forms of the vectors are therefore

$$\begin{aligned}\mathbf{v}_s &= 5\cos(-140^\circ)\mathbf{i} + 5\sin(-140^\circ)\mathbf{j}\\ &\simeq -3.830\mathbf{i} - 3.214\mathbf{j},\end{aligned}$$

$$\begin{aligned}\mathbf{v}_c &= 2\cos(130^\circ)\mathbf{i} + 2\sin(130^\circ)\mathbf{j}\\ &\simeq -1.286\mathbf{i} + 1.532\mathbf{j}.\end{aligned}$$

The resultant velocity is

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_s + \mathbf{v}_c\\ &\simeq (-3.830 - 1.286)\mathbf{i} + (-3.214 + 1.532)\mathbf{j}\\ &= -5.116\mathbf{i} - 1.682\mathbf{j}.\end{aligned}$$

The strategy on page 22 may now be applied. The resultant speed of the ship is

$$|\mathbf{v}| \simeq \sqrt{(-5.116)^2 + (-1.682)^2} \simeq 5.4.$$

Since the components of $\mathbf{v}$ are $v_1 \simeq -5.116$, $v_2 \simeq -1.682$, we have

$$\begin{aligned}\phi &\simeq \arctan(|-1.682/(-5.116)|)\\ &= \arctan(1.682/5.116)\\ &\simeq 18^\circ.\end{aligned}$$

Also, $(-5.116, -1.682)$ lies in the third quadrant, so the direction of $\mathbf{v}$ is $\theta = -(180^\circ - \phi) \simeq -162^\circ$. This corresponds to the bearing S 72°W.

Thus the resultant velocity of the ship is $5.4\,\mathrm{m\,s^{-1}}$ at S 72° W.

Solution 3.1

The angle C is given by

$$C = 180^\circ - 87^\circ - 63^\circ = 30^\circ.$$

Using the Sine Rule in the form

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

gives

$$a = \frac{c\sin A}{\sin C} = \frac{3.2\sin(87^\circ)}{\sin(30^\circ)} = 6.4 \quad \text{(to 1 d.p.)}.$$

Hence $a = 6.4$ to one decimal place.

Solution 3.2

Using the Sine Rule in the form

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

we have

$$\sin A = \frac{a\sin B}{b} = \frac{2.7\sin(35^\circ)}{5.1} \simeq 0.3037.$$

This gives the possible solutions

$$A \simeq \arcsin(0.3037) \simeq 17.7^\circ$$

and

$$A \simeq 180^\circ - 17.7^\circ = 162.3^\circ.$$

However, since $a < b$, we can deduce that $A < B = 35^\circ$, which rules out the second possible solution. Hence we have $A \simeq 17.7^\circ$.

Solution 3.3

The largest angle is opposite the longest side, so it is the angle B which is required. Applying the Cosine Rule in the form

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

we obtain

$$\cos B = \frac{9^2 + 10^2 - 12^2}{2 \times 9 \times 10} \simeq 0.2056,$$

so

$$B \simeq \arccos(0.2056) = 78.1^\circ \quad \text{(to 1 d.p.)}.$$

Hence the largest angle of the triangle is 78.1° to one decimal place.

Solution 3.4

First apply the Cosine Rule in the form

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

This gives

$$a^2 = 8^2 + 15^2 - 2 \times 8 \times 15\cos(60^\circ) = 169,$$

so $a = 13$.

Now apply the Sine Rule, in the form

$$\frac{\sin B}{b} = \frac{\sin A}{a},$$

to obtain

$$\sin B = \frac{b \sin A}{a} = \frac{8 \sin(60^\circ)}{13} \simeq 0.5329$$

Hence, since $b < a$ and so $B < A$, we have

$$B = 32.2^\circ \quad \text{(to 1 d.p.)}.$$

Finally, use the angle sum formula to find

$$C = 180^\circ - 60^\circ - 32.2^\circ = 87.8^\circ \quad \text{(to 1 d.p.)}.$$

Hence, in $\triangle ABC$, we have $a = 13$, $b = 8$, $c = 15$, $A = 60^\circ$, $B = 32.2^\circ$ and $C = 87.8^\circ$, all to one decimal place.

Solution 3.5

As before, let $\mathbf{a}$ be the displacement from Exeter to Belfast, and let $\mathbf{b}$ be the displacement from Belfast to Edinburgh. The Triangle Rule diagram from Solution 2.6 is repeated in Figure S.12, together with a separate triangle that shows the given side-length and angle information.

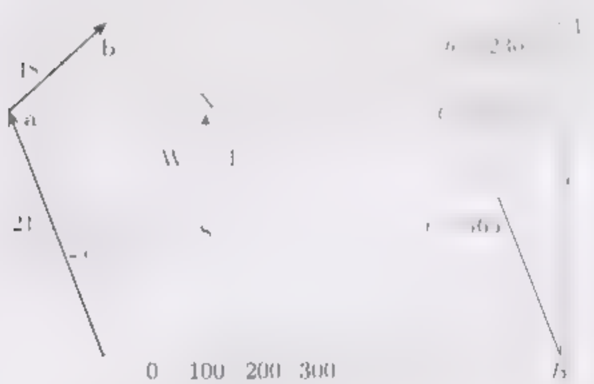


Figure S.12

We know that $a = 465$, $b = 230$ and $C = 180^\circ - (48^\circ + 21^\circ) = 111^\circ$. We wish to find c (the magnitude of $\mathbf{a} + \mathbf{b}$) and B (since the bearing for $\mathbf{a} + \mathbf{b}$ is either N $(21^\circ - B)$ W or N $(B - 21^\circ)$ E).

Using the Cosine Rule in the form

$$c^2 = a^2 + b^2 - 2ab \cos C,$$

we obtain

$$\begin{aligned} c^2 &= 465^2 + 230^2 - 2 \times 465 \times 230 \cos(111^\circ) \\ &\simeq 345\,779\ 90, \end{aligned}$$

so $c = 588$ (to the nearest km).

To find the angle B we now apply the Sine Rule, in the form

$$\frac{\sin B}{b} = \frac{\sin C}{c}.$$

This gives

$$\sin B = \frac{b \sin C}{c} \simeq \frac{230 \sin(111^\circ)}{588} \simeq 0.3652$$

Since $b < c$, and hence $B < C$, we find B 21° (to the nearest degree).

Hence the displacement from Exeter to Edinburgh is 588 km due North. (As expected, this agrees with the result found using components in Solution 2.6.)

Solution 3.6

As before, let $\mathbf{v}_s$ be the velocity of the ship in still water, and let $\mathbf{v}_c$ be the velocity of the current. The Triangle Rule diagram from Solution 2.8 is repeated in Figure S.13, together with a separate triangle that shows the given side-length and angle information.

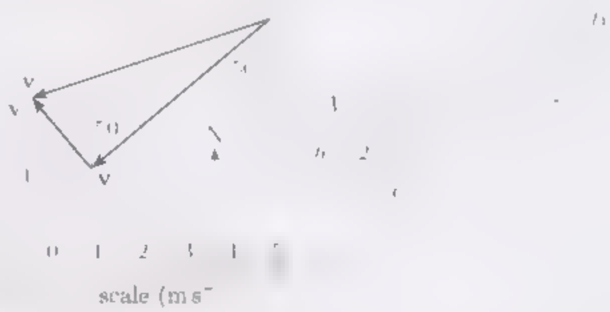


Figure S.13

We know that $a = 5$, $b = 2$ and $C = 40^\circ + 50^\circ = 90$. We wish to find c (the magnitude of $\mathbf{v} = \mathbf{v}_s + \mathbf{v}_c$) and B (since the bearing for $\mathbf{v}$ is S $(50^\circ + B)$ W).

Since the triangle is right-angled, Pythagoras' Theorem gives

$$c^2 = a^2 + b^2 = 5^2 + 2^2 = 29,$$

so $c = 5.4$ (to 1 d.p.).

The angle B is given by $\arcsin(2/c)$ (or by arctan from which we find that $B = 22^\circ$ (to the nearest degree).

Hence the resultant velocity of the ship is $5.4\,\mathrm{m\,s^{-1}}$ at S 72° W (in agreement with Solution 2.8).

Solution 4.1

Since the object remains at rest, the Equilibrium Condition applies; that is,

$$\mathbf{F} + \mathbf{G} + \mathbf{H} = \mathbf{0}$$

Hence we find that

$$\begin{aligned} \mathbf{F} &= -\mathbf{G} - \mathbf{H} \\ &= -(2\mathbf{i} + \mathbf{j}) - (-\mathbf{j}) \\ &= -2\mathbf{i} \end{aligned}$$

Solution 4.2

The force diagram and corresponding triangle of forces are shown in Figure S.14. (The anticlockwise version of the triangle of forces is equally acceptable.)

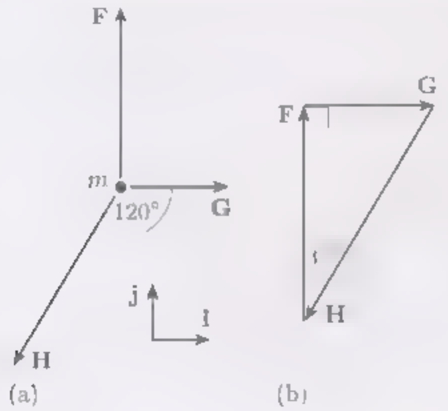


Figure S.14

All of the angles in the triangle of forces can be found from the given force directions, as indicated in Figure S.14(b). Since the triangle of forces is right-angled, we have

$$|\mathbf{G}| = |\mathbf{F}| \tan(30^\circ) = |\mathbf{H}| \sin(30^\circ),$$

so the required force magnitudes are given by

$$|\mathbf{F}| = \frac{|\mathbf{G}|}{\tan(30^\circ)} = \frac{70}{1/\sqrt{3}} = 70\sqrt{3}\,\mathrm{N},$$

$$|\mathbf{H}| = \frac{|\mathbf{G}|}{\sin(30^\circ)} = \frac{70}{1/2} = 140\,\mathrm{N}.$$

Solution 5.1

The force diagram is shown in Figure S.15. (See Frame 3 for definitions of the forces.)

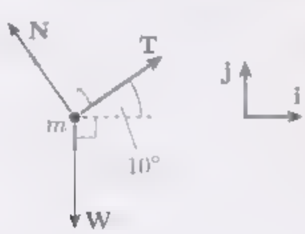


Figure S.15

In component form, the three forces are

$$\mathbf{T} = |\mathbf{T}| \cos(10^\circ)\mathbf{i} + |\mathbf{T}| \sin(10^\circ)\mathbf{j},$$
$$\mathbf{N} = |\mathbf{N}| \cos(100^\circ)\mathbf{i} + |\mathbf{N}| \sin(100^\circ)\mathbf{j},$$
$$\mathbf{W} = -|\mathbf{W}|\mathbf{j} = -mg\mathbf{j} = -26\,000\mathbf{j}.$$

On applying the Equilibrium Condition, $\mathbf{T} + \mathbf{N} + \mathbf{W} = \mathbf{0}$, we obtain

$$(|\mathbf{T}| \cos(10^\circ) + |\mathbf{N}| \cos(100^\circ))\mathbf{i}$$
$$+ (|\mathbf{T}| \sin(10^\circ) + |\mathbf{N}| \sin(100^\circ) - 26\,000)\mathbf{j} = \mathbf{0}.$$

Looking in turn at the **i**- and **j**-components gives

$$|\mathbf{T}| \cos(10^\circ) + |\mathbf{N}| \cos(100^\circ) = 0,$$
$$|\mathbf{T}| \sin(10^\circ) + |\mathbf{N}| \sin(100^\circ) = 26\,000$$

This pair of simultaneous equations can be solved in several different ways. For example, using trigonometric identities from Chapter A2, we have

$$\cos(100^\circ) = \sin(90^\circ - 100^\circ) = -\sin(10^\circ),$$
$$\sin(100^\circ) = \cos(90^\circ - 100^\circ) = \cos(10^\circ),$$

so the simultaneous equations can be written as

$$|\mathbf{T}| \cos(10^\circ) - |\mathbf{N}| \sin(10^\circ) = 0,$$
$$|\mathbf{T}| \sin(10^\circ) + |\mathbf{N}| \cos(10^\circ) = 26\,000.$$

On adding $\sin(10^\circ)$ times the second equation to $\cos(10^\circ)$ times the first, we obtain (since $\cos^2(10^\circ) + \sin^2(10^\circ) = 1$)

$$|\mathbf{T}| = 26\,000 \sin(10^\circ) \simeq 4515\,\mathrm{N}.$$

From here on, the working is as shown in Frame 6.

Solution 5.2

Choose **i** to represent 1 N horizontally and **j** to represent 1 N vertically upwards. The force diagram is shown in Figure S.16. (See Frame 9 for definitions of the forces.)

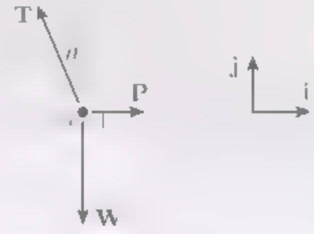


Figure S.16

In component form, the three forces are

$$\mathbf{P} = |\mathbf{P}|\mathbf{i} = 350\mathbf{i},$$
$$\mathbf{W} = -|\mathbf{W}|\mathbf{j} = -mg\mathbf{j} = -20\,000\mathbf{j},$$
$$\mathbf{T} = |\mathbf{T}| \cos(90^\circ + \theta)\mathbf{i} + |\mathbf{T}| \sin(90^\circ + \theta)\mathbf{j}$$
$$= -|\mathbf{T}| \sin\theta\,\mathbf{i} + |\mathbf{T}| \cos\theta\,\mathbf{j}.$$

On applying the Equilibrium Condition, $\mathbf{P} + \mathbf{W} + \mathbf{T} = \mathbf{0}$, we obtain

$$(350 - |\mathbf{T}|\sin\theta)\mathbf{i} + (-20\,000 + |\mathbf{T}|\cos\theta)\mathbf{j} = \mathbf{0}.$$

Looking in turn at the $\mathbf{i}$- and $\mathbf{j}$-components gives

$$|\mathbf{T}|\sin\theta = 350, \quad |\mathbf{T}|\cos\theta = 20\,000,$$

so, on dividing the first equation by the second, we obtain

$$\tan\theta = \frac{350}{20\,000} = 0.0175.$$

The solution now proceeds as shown in Frame 10.

Solution 5.3

Let $\mathbf{W}$ be the weight of the flower basket, let $\mathbf{T}_1$ be the tension from the angled cord, and let $\mathbf{T}_2$ be the tension from the horizontal cord. The force diagram is shown in Figure S.17.

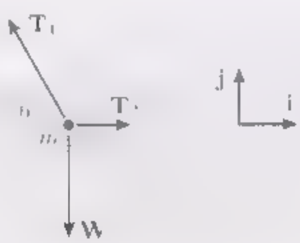


Figure S.17

(a) Choose $\mathbf{i}$ to represent 1 N horizontally and $\mathbf{j}$ to represent 1 N vertically upwards. Then the component forms of the three forces are

$$\begin{aligned}\mathbf{W} &= -|\mathbf{W}|\mathbf{j} = -mg\mathbf{j} = -40\mathbf{j},\\ \mathbf{T}_1 &= |\mathbf{T}_1|\cos(120^\circ)\mathbf{i} + |\mathbf{T}_1|\sin(120^\circ)\mathbf{j}\\ &= -|\mathbf{T}_1|\cos(60^\circ)\mathbf{i} + |\mathbf{T}_1|\sin(60^\circ)\mathbf{j}\\ &= -\tfrac{1}{2}|\mathbf{T}_1|\mathbf{i} + \tfrac{1}{2}\sqrt{3}|\mathbf{T}_1|\mathbf{j},\\ \mathbf{T}_2 &= |\mathbf{T}_2|\mathbf{i}.\end{aligned}$$

On applying the Equilibrium Condition, $\mathbf{W} + \mathbf{T}_1 + \mathbf{T}_2 = \mathbf{0}$, we obtain

$$(-\tfrac{1}{2}|\mathbf{T}_1| + |\mathbf{T}_2|)\mathbf{i} + (-40 + \tfrac{1}{2}\sqrt{3}|\mathbf{T}_1|)\mathbf{j} = \mathbf{0}$$

which is equivalent to

$$|\mathbf{T}_2| = \tfrac{1}{2}|\mathbf{T}_1| \quad \text{and} \quad 40 = \tfrac{1}{2}\sqrt{3}|\mathbf{T}_1|.$$

It follows that

$$\begin{aligned}|\mathbf{T}_1| &= \frac{80}{\sqrt{3}} \simeq 46.2,\\ |\mathbf{T}_2| &= \tfrac{1}{2}|\mathbf{T}_1| \simeq 23.1.\end{aligned}$$

Hence the tensions have magnitude 46.2 N from the angled cord and 23.1 N from the horizontal cord.

(b) The triangle of forces is shown in Figure S.18.

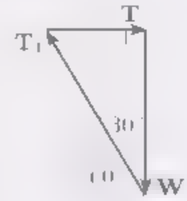


Figure S.18

Since this triangle is right-angled, we have

$$\begin{aligned}|\mathbf{T}_1| &= \frac{|\mathbf{W}|}{\cos(30^\circ)} = \frac{40}{\frac{1}{2}\sqrt{3}} \simeq 46.2,\\ |\mathbf{T}_2| &= |\mathbf{W}|\tan(30^\circ) = \frac{40}{\sqrt{3}} \simeq 23.1,\end{aligned}$$

which provide the same answers as in part (a).

Solution 5.4

Denote the weight of the trolley by $\mathbf{W}$ and the normal reaction from the ramp by $\mathbf{N}$. The force diagram and corresponding triangle of forces are shown in Figure S.19.

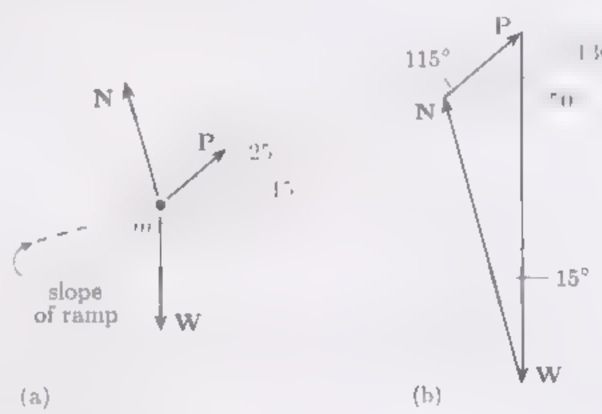


Figure S.19

All of the angles in the triangle of forces can be found from the given force directions, as indicated in Figure S.19(b). So the Sine Rule can be applied directly, in the form

$$\frac{|\mathbf{P}|}{\sin(15^\circ)} = \frac{|\mathbf{W}|}{\sin(115^\circ)}.$$

Hence the required pulling force magnitude is given by

$$\begin{aligned}|\mathbf{P}| &= \frac{|\mathbf{W}|\sin(15^\circ)}{\sin(115^\circ)}\\ &= \frac{60 \times 10 \times \sin(15^\circ)}{\sin(115^\circ)}\\ &\simeq 171\ \text{N}.\end{aligned}$$

Solutions to Exercises

Solution 1.1

(a) The column forms for the three vectors are

$$\mathbf{a} = \begin{pmatrix} -7 \\ 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

(b) The component form of $\mathbf{a} + 3\mathbf{b} - \mathbf{c}$ is

$$\begin{aligned}&(-7\mathbf{i} + 3\mathbf{j}) + 3(5\mathbf{i} - 2\mathbf{j}) - (3\mathbf{j})\\ &= (-7 + 3 \times 5 + 0)\mathbf{i} + (3 + 3 \times (-2) - 3)\mathbf{j}\\ &= 8\mathbf{i} - 6\mathbf{j},\end{aligned}$$

which has $\mathbf{i}$-component 8 and $\mathbf{j}$-component -6.

Solution 1.2

A labelled arrow to represent each of the vectors appears in Figure S.20. The Triangle Rule is used to obtain the arrows for $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$.

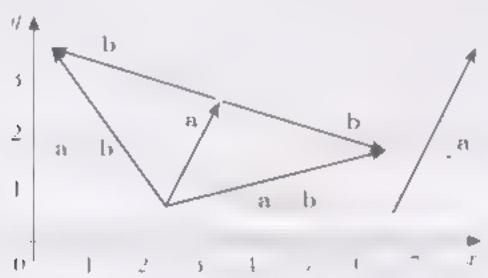


Figure S.20

Solution 2.1

Applying the equation $\mathbf{a} = |\mathbf{a}|\cos\theta\,\mathbf{i} + |\mathbf{a}|\sin\theta\,\mathbf{j}$, where $|\mathbf{a}| = 7$ and $\theta = -70°$, we obtain the component form

$$\begin{aligned}\mathbf{a} &= 7\cos(-70°)\mathbf{i} + 7\sin(-70°)\mathbf{j}\\ &\simeq 2.39\mathbf{i} - 6.58\mathbf{j} \quad \text{(to 2 d.p.)}.\end{aligned}$$

Solution 2.2

In each case, the strategy on page 22 may be applied.

The magnitude of the vector $\mathbf{a} = -3\mathbf{i} + 2\mathbf{j}$ is

$$|\mathbf{a}| = \sqrt{(-3)^2 + 2^2} = \sqrt{13} \simeq 3.6.$$

Since the components are $a_1 = -3$, $a_2 = 2$, we have

$$\phi = \arctan(|2/(-3)|) = \arctan\tfrac{2}{3} \simeq 33.7°.$$

Also, $(-3, 2)$ lies in the second quadrant, so the direction of $\mathbf{a}$ is $\theta = 180° - \phi \simeq 146.3°$.

The magnitude of the vector $\mathbf{b} = 6\mathbf{i} - \mathbf{j}$ is

$$|\mathbf{b}| = \sqrt{6^2 + (-1)^2} = \sqrt{37} \simeq 6.1.$$

Since the components are $b_1 = 6$, $b_2 = -1$, we have

$$\phi = \arctan(|-1/6|) = \arctan\tfrac{1}{6} \simeq 9.5°.$$

Also, $(6, -1)$ lies in the fourth quadrant, so the direction of $\mathbf{b}$ is $\theta = -\phi \simeq -9.5°$.

The sum of the vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{c} = \mathbf{a} + \mathbf{b} = (-3 + 6)\mathbf{i} + (2 - 1)\mathbf{j} = 3\mathbf{i} + \mathbf{j}.$$

The magnitude of this vector is

$$|\mathbf{c}| = \sqrt{3^2 + 1^2} = \sqrt{10} \simeq 3.2.$$

Since the components are $c_1 = 3$, $c_2 = 1$, we have

$$\phi = \arctan\tfrac{1}{3} \simeq 18.4°.$$

Also, $(3, 1)$ lies in the first quadrant, so the direction of $\mathbf{c}$ is $\theta = \phi \simeq 18.4°$.

Solution 2.3

(a) The displacement from Leicester to Derby is the opposite of the given displacement from Derby to Leicester, so it is 32 km at N 45°W.

(b) Let $\mathbf{i}$ be 1 km East and let $\mathbf{j}$ be 1 km North. Denote the displacement from Leicester to Derby by the vector $\mathbf{a}$, and the displacement from Derby to Birmingham by the vector $\mathbf{b}$. Then the required displacement from Leicester to Birmingham is represented by the vector $\mathbf{a} + \mathbf{b}$. These vectors are sketched in Figure S.21.

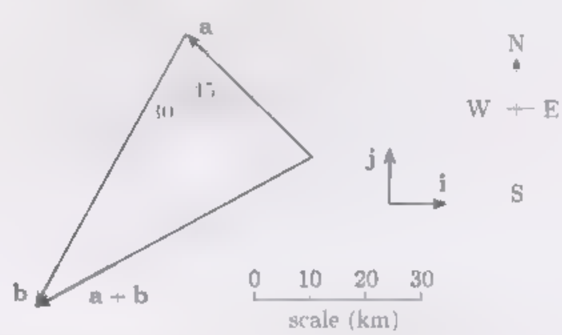


Figure S.21

The vector $\mathbf{a}$ has magnitude $|\mathbf{a}| = 32$ and direction $90° + 45° = 135°$, so its component form is

$$\begin{aligned}\mathbf{a} &= 32\cos(135°)\mathbf{i} + 32\sin(135°)\mathbf{j}\\ &\simeq -22.63\mathbf{i} + 22.63\mathbf{j}.\end{aligned}$$

The vector $\mathbf{b}$ has magnitude $|\mathbf{b}| = 57$ and direction $-(90° + 30°) = -120°$, so its component form is

$$\begin{aligned}\mathbf{b} &= 57\cos(-120°)\mathbf{i} + 57\sin(-120°)\mathbf{j}\\ &\simeq -28.50\mathbf{i} - 49.36\mathbf{j}.\end{aligned}$$

The resultant is

$$\begin{aligned} \mathbf{c} &= \mathbf{a} + \mathbf{b} \\ &\simeq (-22.63 - 28.50)\mathbf{i} + (22.63 - 49.36)\mathbf{j} \\ &= -51.13\mathbf{i} - 26.73\mathbf{j} \end{aligned}$$

The strategy on page 22 may now be applied. The vector $\mathbf{c}$ has magnitude

$$c \simeq \sqrt{(-51.13)^2 + (-26.73)^2} \simeq 57.7.$$

Since the components are $c_1 \simeq -51.13$, $c_2 \simeq -26.73$, we have

$$\begin{aligned} \phi &\simeq \arctan(|-26.73/(-51.13)|) \\ &= \arctan(26.73/51.13) \\ &\simeq 27.6^\circ. \end{aligned}$$

Also, $(-51.13, -26.73)$ lies in the third quadrant, so the direction of $\mathbf{c}$ is $\theta = -(180^\circ - \phi) \simeq -152.4^\circ$. This corresponds to the bearing S 62.4° W.

Hence the displacement from Leicester to Birmingham is 57.7 km at S 62.4° W.

Solution 2.4

Let $\mathbf{i}$ be $1\,\mathrm{m\,s^{-1}}$ East and let $\mathbf{j}$ be $1\,\mathrm{m\,s^{-1}}$ North. Suppose that $\mathbf{v}_a$ is the velocity of the aeroplane in still air and $\mathbf{v}_w$ is the velocity of the wind. These vectors are shown in Figure S.22.

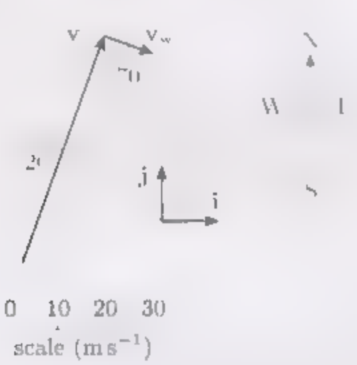


Figure S.22

From the information given, $\mathbf{v}_a$ has magnitude $|\mathbf{v}_a| = 50$ and direction $\theta_a = 70^\circ$.

The wind comes *from* N 70° W and hence blows *towards* S 70° E, for which the direction is $-(90^\circ - 70^\circ)$. Hence the vector $\mathbf{v}_w$ has magnitude $|\mathbf{v}_w| = 10$ and direction $\theta_w = -20^\circ$.

The component forms of the vectors are therefore

$$\begin{aligned} \mathbf{v}_a &= 50\cos(70^\circ)\mathbf{i} + 50\sin(70^\circ)\mathbf{j} \\ &\simeq 17.10\mathbf{i} + 46.98\mathbf{j}, \end{aligned}$$

$$\begin{aligned} \mathbf{v}_w &= 10\cos(-20^\circ)\mathbf{i} + 10\sin(-20^\circ)\mathbf{j} \\ &\simeq 9.40\mathbf{i} - 3.42\mathbf{j}. \end{aligned}$$

The resultant velocity is

$$\begin{aligned} \mathbf{v} &= \mathbf{v}_a + \mathbf{v}_w \\ &\simeq (17.10 + 9.40)\mathbf{i} + (46.98 - 3.42)\mathbf{j} \\ &= 26.50\mathbf{i} + 43.56\mathbf{j} \end{aligned}$$

The strategy on page 22 may now be applied. The resultant speed of the aeroplane is

$$|\mathbf{v}| \simeq \sqrt{(26.50)^2 + (43.56)^2} \simeq 51.0.$$

Since the components of $\mathbf{v}$ are $v_1 \simeq 26.50$, $v_2 \simeq 43.56$, we have

$$\phi \simeq \arctan(43.56/26.50) \simeq 58.7^\circ.$$

Also, $(26.50, 43.56)$ lies in the first quadrant, so the direction of $\mathbf{v}$ is $\theta = \phi \simeq 58.7^\circ$. This corresponds to the bearing N 31.3° E.

Thus the velocity of the aeroplane relative to the ground is $51.0\,\mathrm{m\,s^{-1}}$ at N 31.3° E.

Solution 3.1

We have $a = 12$, $B = 59^\circ$ and $C = 73^\circ$. The third angle of the triangle is

$$A = 180^\circ - 59^\circ - 73^\circ = 48^\circ$$

Using the Sine Rule in the form

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

we obtain

$$b = \frac{a\sin B}{\sin A} = \frac{12\sin(59^\circ)}{\sin(48^\circ)} = 13.8 \quad \text{(to 1 d.p.)},$$

$$c = \frac{a\sin C}{\sin A} = \frac{12\sin(73^\circ)}{\sin(48^\circ)} = 15.4 \quad \text{(to 1 d.p.)}.$$

Hence, in $\triangle ABC$, we have $a = 12$, $b = 13.8$, $c = 15.4$, $A = 48^\circ$, $B = 59^\circ$ and $C = 73^\circ$.

Solution 3.2

The largest and smallest angles of the triangle are respectively C and A, since these are opposite the longest side c and shortest side a. Applying the Cosine Rule in the form

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

we obtain

$$\cos C = \frac{4^2 + 5^2 - 6^2}{2 \times 4 \times 5} = 0.125,$$

so

$$C = \arccos(0.125) \simeq 82.819\,244\,22^\circ.$$

Applying the Cosine Rule once more, in the form

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

we obtain

$$\cos A = \frac{5^2 + 6^2 - 4^2}{2 \times 5 \times 6} = 0.75,$$

so

$$A = \arccos(0.75) \simeq 41.409\,622\,11^\circ.$$

(The Sine Rule could also have been used to obtain the value of A.)

Hence the largest angle of the triangle is $C = 82.819\,244\,22^\circ$, and the smallest angle is $A = 41.409\,622\,11^\circ$.

To the accuracy of the calculator, we have $C = 2A$. (This is, in fact, an exact result: the largest angle in a 4-5-6 triangle is twice the size of the smallest.)

Solution 3.3

As before, let $\mathbf{v}_a$ be the velocity of the aeroplane in still air, and let $\mathbf{v}_w$ be the velocity of the wind. The Triangle Rule diagram from the solution to Exercise 2.4 is repeated in Figure S.23, together with a separate triangle that shows the given side-length and angle information.

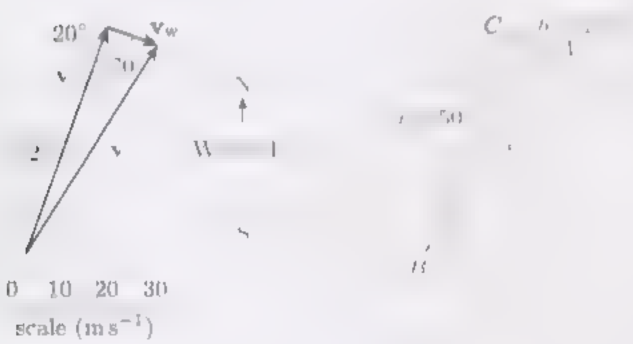


Figure S.23

We know that $a = 50$, $b = 10$ and $C = 20^\circ + 70^\circ = 90^\circ$. We wish to find c (the magnitude of $\mathbf{v} = \mathbf{v}_a + \mathbf{v}_w$) and B (since the bearing for $\mathbf{v}$ is N$\,(20^\circ + B)\,$E).

Since the triangle is right-angled, Pythagoras' Theorem gives

$$c^2 = a^2 + b^2 = 50^2 + 10^2 = 2600,$$

so $c = 51.0$ (to 1 d.p.).

The angle B is given by $\arcsin(10/c)$ (or by $\arctan \frac{1}{5}$), from which we find that $B = 11.3^\circ$ (to 1 d.p.).

Hence the velocity of the aeroplane relative to the ground is $51.0\,\mathrm{m\,s^{-1}}$ at N 31.3°E (in agreement with the solution to Exercise 2.4).

Solution 3.4

Let $\mathbf{v}_a$ be the velocity of the aeroplane in still air, let $\mathbf{v}_w$ be the velocity of the wind, and let $\mathbf{v}$ be their resultant. The Triangle Rule diagram for the specified situation is shown in Figure S.24, together with a separate triangle that shows the given side-length and angle information.

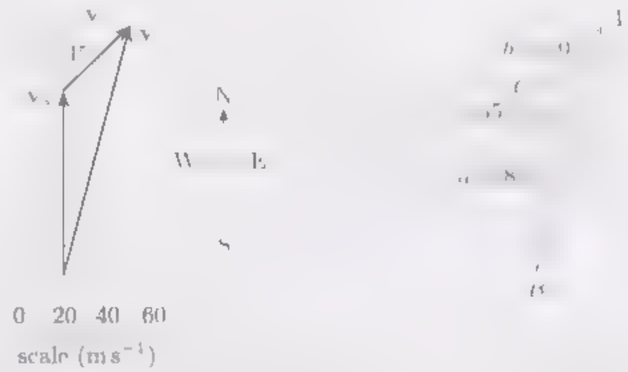


Figure S.24

We know that $a = 80$, $b = 40$ and $C = 180^\circ - 45^\circ = 135^\circ$. We wish to find c (the magnitude of $\mathbf{v}$) and B (since the bearing for $\mathbf{v}$ is N$\,(B)\,$E).

Using the Cosine Rule in the form

$$c^2 = a^2 + b^2 - 2ab\cos C,$$

we obtain

$$c^2 = 80^2 + 40^2 - 2 \times 80 \times 40 \cos(135^\circ)$$
$$\simeq 12\,525.5,$$

so $c = 111.9$ (to 1 d.p.).

Now apply the Sine Rule, in the form

$$\frac{\sin B}{b} = \frac{\sin C}{c},$$

to obtain

$$\sin B = \frac{b \sin C}{c} \simeq \frac{40 \sin(135^\circ)}{111.9} \simeq 0.2527.$$

Since $b < c$, and hence $B < C$, we find $B = 14.6^\circ$ (to 1 d.p.).

Hence the velocity of the aeroplane relative to the ground is $111.9\,\mathrm{m\,s^{-1}}$ at N 14.6°E.

Solution 4.1

Since the object remains at rest, the Equilibrium Condition applies; that is,

$$\mathbf{P} + \mathbf{Q} + \mathbf{R} = \mathbf{0}.$$

Hence we find that

$$\begin{aligned}\mathbf{R} &= -\mathbf{P} - \mathbf{Q} \\ &= -(3\mathbf{i} - 6\mathbf{j}) - (5\mathbf{i} + 2\mathbf{j}) \\ &= -8\mathbf{i} + 4\mathbf{j}.\end{aligned}$$

Solution 4.2

The force diagram and corresponding triangle of forces are shown in Figure S.25.

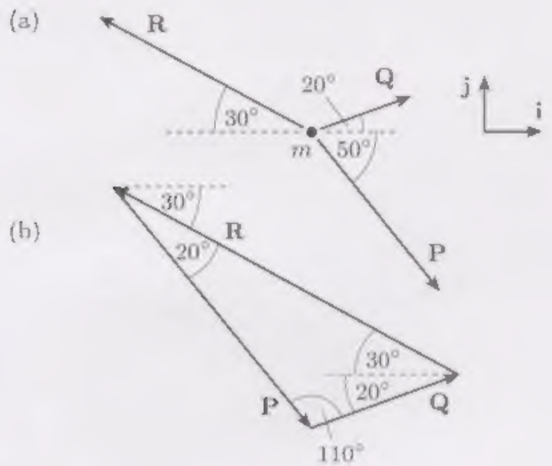


Figure S.25

All of the angles in the triangle of forces can be found from the given force directions, as indicated in Figure S.25(b), so the Sine Rule can be applied directly, in the form

$$\frac{|\mathbf{P}|}{\sin(50^\circ)} = \frac{|\mathbf{Q}|}{\sin(20^\circ)} = \frac{|\mathbf{R}|}{\sin(110^\circ)}.$$

Hence the required force magnitudes are given by

$$|\mathbf{Q}| = \frac{|\mathbf{P}|\sin(20^\circ)}{\sin(50^\circ)} \simeq 53.6\,\text{N},$$

$$|\mathbf{R}| = \frac{|\mathbf{P}|\sin(110^\circ)}{\sin(50^\circ)} \simeq 147.2\,\text{N}.$$

Solution 5.1

Let $\mathbf{W}$ be the weight of the block, let $\mathbf{N}$ be the normal reaction, and let $\mathbf{P}$ be the horizontal pushing force. The force diagram is shown in Figure S.26.

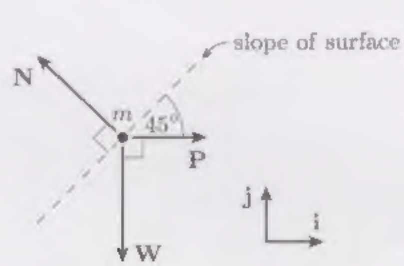


Figure S.26

(a) Choose $\mathbf{i}$ to represent 1 N horizontally and $\mathbf{j}$ to represent 1 N vertically upwards. Then the component forms of the three forces are

$$\mathbf{W} = -|\mathbf{W}|\mathbf{j} = -mg\mathbf{j} = -6\mathbf{j},$$

$$\begin{aligned}\mathbf{N} &= |\mathbf{N}|\cos(135^\circ)\mathbf{i} + |\mathbf{N}|\sin(135^\circ)\mathbf{j}\\ &= -|\mathbf{N}|\cos(45^\circ)\mathbf{i} + |\mathbf{N}|\sin(45^\circ)\mathbf{j}\\ &= -\tfrac{1}{2}\sqrt{2}\,|\mathbf{N}|\mathbf{i} + \tfrac{1}{2}\sqrt{2}\,|\mathbf{N}|\mathbf{j},\end{aligned}$$

$$\mathbf{P} = |\mathbf{P}|\mathbf{i}.$$

On applying the Equilibrium Condition, $\mathbf{W} + \mathbf{N} + \mathbf{P} = \mathbf{0}$, we obtain

$$(-\tfrac{1}{2}\sqrt{2}\,|\mathbf{N}| + |\mathbf{P}|)\mathbf{i} + (-6 + \tfrac{1}{2}\sqrt{2}\,|\mathbf{N}|)\mathbf{j} = \mathbf{0},$$

which is equivalent to

$$|\mathbf{P}| = \tfrac{1}{2}\sqrt{2}\,|\mathbf{N}| \quad \text{and} \quad 6 = \tfrac{1}{2}\sqrt{2}\,|\mathbf{N}|.$$

It follows that

$$|\mathbf{N}| = 6\sqrt{2} \quad \text{and} \quad |\mathbf{P}| = \tfrac{1}{2}\sqrt{2}\,|\mathbf{N}| = 6.$$

Hence the magnitude of the pushing force is 6 N (the same as the magnitude of the weight).

(b) The force diagram and corresponding triangle of forces are shown in Figure S.27.

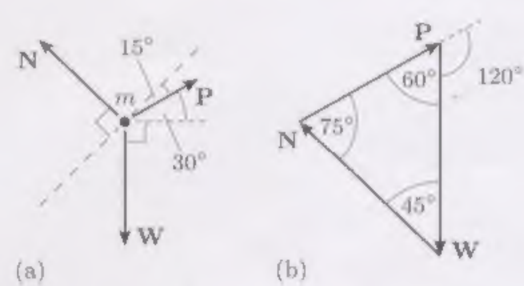


Figure S.27

Applying the Sine Rule in the form

$$\frac{|\mathbf{P}|}{\sin(45^\circ)} = \frac{|\mathbf{W}|}{\sin(75^\circ)},$$

we find that the magnitude of the pushing force is now

$$|\mathbf{P}| = \frac{|\mathbf{W}|\sin(45^\circ)}{\sin(75^\circ)} = \frac{6\sin(45^\circ)}{\sin(75^\circ)} \simeq 4.4\,\text{N}.$$

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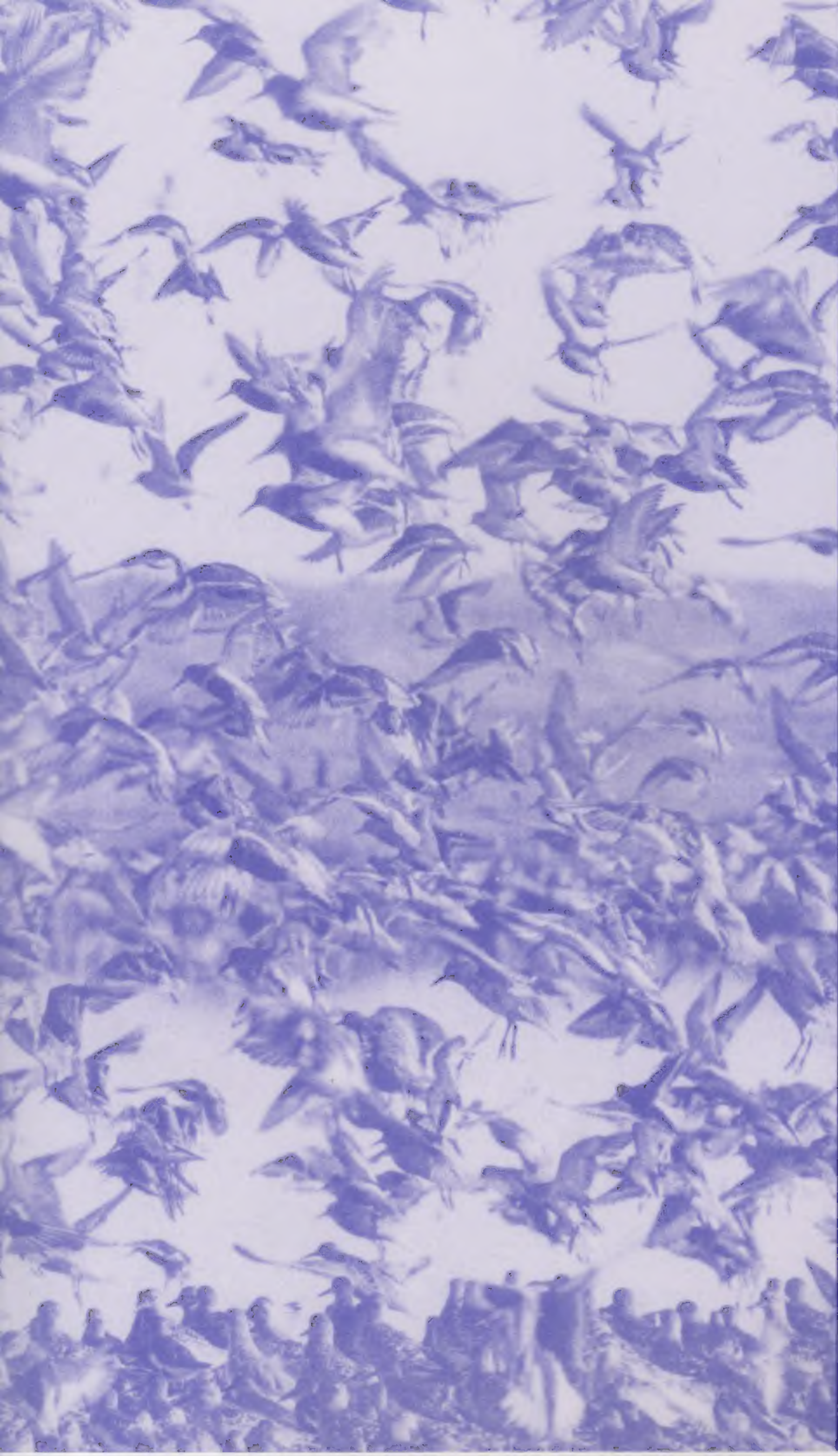

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